Random matrices and maps

Thomas Buc–d'Alché

ENS DE LYON

THÈSE

en vue de l'obtention du grade de Docteur, délivré par

l'ÉCOLE NORMALE SUPERIEURE DE LYON

École Doctorale N° 512

InfoMaths - Informatique et Mathématiques de Lyon

Discipline : MATHEMATIQUES (Mathématiques)

Soutenue publiquement le 08/07/2025, par :

Thomas Buc–d'Alché

Matrices et cartes aléatoires

Random matrices and maps

Après avis de :

Guillaume Chapuy, Directeur de recherche, Université Paris Cité	Rapporteur
Benoît Collins, Professeur, Kyoto University	Rapporteur

Devant le jury composé de :

Guillaume Chapuy, Directeur de recherche, Université Paris Cité	Rapporteur
Benoît Collins, Professeur, Kyoto University	Rapporteur
Nalini Anantharaman, Professeure, Université de Strasbourg	Examinatrice
Jérémie Bouttier, Professeur des universités, Sorbonne Université	Examinateur
Guillaume Dubach, Professeur Monge, Ecole Polytechnique	Examinateur
Elba Garcia-Failde, Maîtresse de conférences, Sorbonne Université	Examinatrice
Alice Guionnet, Directrice de recherche, ENS de Lyon	Directrice de thèse
Grégory Miermont, Professeur des universités, ENS de Lyon	Co-directeur de thèse

Résumé

Cette thèse traite de plusieurs problèmes reliés aux matrices aléatoires et à l'énumération de cartes. Informellement, les cartes sont des graphes dessinés sur des surfaces. Les travaux des physiciens Brézin, Itzkson, Parisi, et Zuber ont permis de comprendre que le problème de l'énumération des cartes est relié à la distribution des valeurs propres de matrices aléatoires Hermitiennes Gaussiennes. Ce lien, beaucoup étudié depuis, s'est révélé fructueux dans les deux directions: une bonne compréhension de la combinatoire des cartes permet de décrire le spectre de matrices aléatoires, et des méthodes analytiques applicables aux intégrales de matrices permettent d'approcher des problèmes combinatoires a priori difficiles. Le Chapitre 3 de cette thèse propose une description de modèles de matrices aléatoires unitaires en terme d'une famille de cartes, les cartes de type unitaire. Ces cartes constituent une généralisation d'une famille d'objets combinatoires liés aux probabilités libres, les nombres de Hurwitz monotones. D'autre part, la distribution de valeurs propres de matrices Hermitiennes unitairement invariantes est un cas particulier d'une famille de mesures appelée β-ensemble. Au Chapitre 4, on propose une méthode directe de calcul des moments du β-ensemble en terme de cartes. Cette approche propose un point de vue nouveau sur les moments du β-ensemble, différent de celui considéré par LaCroix, dans le cadre la b-conjecture de Goulden et Jackson en combinatoire algébrique.

Des relations clés pour étudier les liens entre cartes et matrices aléatoires sont les équations de Dyson-Schwinger. En théorie des matrices aléatoires, ces équations apparaissent comme conséquence de l'invariance par translation de la mesure de référence considérée. Au dela, ces équations apparaissent sous d'autres formes dans de nombreux domaines: en particulier, il s'agit des équations de Tutte en combinatoire des cartes. Elles peuvent être vues comme un cas particulier de la récurrence topologique de Eynard et Orantin. Ecrites pour le β-modèle et dans la limite de grande dimension, les équations de Dyson-Schwinger peuvent être interprétées comme définissant une courbe hyperelliptique, la courbe spectrale. De nombreuses observables de matrices aléatoires correspondent à des objets géométriques définis en terme de la courbe spectrale. Il est alors possible de réinterpréter des identités probabilistes de matrices aléatoires d'un point de vue géométrique. Une telle approche est discutée au Chapitre 5, pour obtenir des analogues Pfaffiens de la formule de Fay.

Plutôt que le spectre, on peut étudier les vecteur propres de matrices aléatoires. Dans ce cadre, on discute du problème de localisation des vecteurs propres d'une matrice d'adjacence d'un graphe aléatoire. Cette question est liée notamment au problème de la localisation d'Anderson, un problème encore partiellement ouvert en physique mathématique. Au Chapitre 6, on étudie le problème de localisation pour le modèle Generalized Random Graph, qui généralise le modèle d'Erdős-Rényi.

Abstract

This thesis discusses several problems related to random matrices and to the enumeration of maps. Informally, a map is a graph drawn on a surface. The work of the physicists Brézin, Itzykson, Parisi, and Zuber showed that the problem of enumerating maps is related to the distribution of eigenvalues of random Gaussian Hermitian matrices. This link, much studied since, turned out to be fruitful in both directions: a good understanding of the combinatorics of maps allows the description of the spectrum of some random matrices, and analytical methods applicable to integrals over spaces of matrices can be used to treat combinatorial problems that are a priori difficult. Chapter 3 of this thesis give a description of random unitary matrix models in terms of a family of maps, the maps of unitary type. These maps are a generalization of a family of combinatorial numbers related to free probability, the monotone Hurwitz numbers. Moreover, the distribution of eigenvalues of unitarily invariant random matrices is a particular case of a family of measure called β-ensemble. In Chapter 4, we give a direct method to compute the moments of the β-ensemble in terms of maps. This approach gives a new point of view on theses moments, different from the one taken by LaCroix in the context of the b-conjecture of Goulden and Jackson in algebraic combinatorics.

Some key relations to study the links between maps and random matrices are the Dyson-Schwinger equations. In random matrix theory, these equations appear as a consequence of the invariance by translation of the relevant reference measure. Beyond this, these equations appear under different guises in many fields: in particular, they are Tutte's equations for the combinatorics of maps. They can be seen as a particular case of the topological recursion of Eynard and Orantin. Written for the β-model, and in the limit of large dimension, the Dyson-Schwinger equations can be interpreted as defining a hyperelliptic curve, the spectral curve. Many observables of random matrix theory correspond to geometric objects defined in terms of the spectral curve. It is then possible to reinterpret probabilistic identities from random matrix theory from a geometric point of view. Such an approach is discussed in Chapter 5, to obtain Pfaffian analogues of Fay's formula.

Rather than the spectrum, one can study the eigenvectors of random matrices. In this context, we discuss the problem of localization of the eigenvectors of the adjacency matrix of a random graph. This question is related in particular to the problem of Anderson localization, still partially open in mathematical physics. In Chapter 6, we study the problem of localization for the Generalized Random Graph model, which generalizes the Erdős-Rényi model.

Remerciements

Je voudrais remercier d'abord Alice et Grégory de m'avoir accompagné pendant cette thèse. J'ai eu la chance d'avoir une *dream team* de directrice et directeur de thèse, qui, chacun avec leur style, m'ont ouvert les portes des mondes des cartes et des matrices aléatoires. Que ce soit avec Grégory, devant un dessin, à échafauder une bijection, ou avec Alice, face à un calcul à qui il faut "donner une chance de marcher", j'ai éprouvé le bonheur que peut procurer la recherche en mathématiques. C'est un magnifique cadeau. Et puis il y a tant d'autres choses: les cours incroyables de Grégory lorsque j'étais élève à l'ENS de Lyon, les conseils stratégiques d'Alice, les relectures innombrables de certains des textes présents dans cette thèse, un financement de thèse[1]... Vraiment, c'était un bon moment!

Je remercie Guillaume Chapuy et Benoît Collins de m'avoir fait l'honneur d'être rapporteurs de ma thèse. J'ai également la chance de compter dans mon jury Jérémie Bouttier, Guillaume Dubach, Elba Garcia-Failde, et Nalini Anantharaman. Je les remercie chaleureusement pour cela. C'est un grand honneur de soutenir ma thèse en présence de tant de chercheuses et chercheurs dont les travaux ont été source d'inspiration pendant ces trois années.

It takes a village to get a PhD... et l'UMPA est un village très accueillant. Je remercie tous ses membres pour avoir développé une si bonne ambiance. Je remercie Virginia, Magalie, et Jessica, sans qui le labo ne tiendrait pas debout, pour leur aide inestimable. Je remercie le personnel de ménage et d'entretien qui rend le labo si agréable à vivre. J'ai eu la chance d'avoir une famille de thèse incroyable: Ronan, Charlie, Vanessa d'un côté, William et Mathieu de l'autre. Je vous suis très reconnaissant pour votre aide, votre amitié, et tant de discussions passionnantes. Merci à mes co-bureaux, Benjamin, Victor, Guillaume, et Alexandre, d'avoir rendu le quotidien si agréable. Merci à Basile, Mehrasa, Hugues, Vianney, Céline, Swann, Mohamed, Alexis, David, Sasha, Slim, Alex, Alex, Alice, Ella, Jana, Rustam, Tanguy, Luca, Luca[2], Claire, et beaucoup d'autres, avec qui j'ai eu la chance de partager tant de beaux moments, autour d'un repas, d'une bière, d'une soirée, d'un film, d'un gateau, d'une chasse aux oeufs, d'un problème de math, ou d'une conférence. Je suis reconnaissant de quitter cette aventure avec autant de nouveaux amis!

Je remercie Thomas, Jean-Christophe, et Nikolay, avec qui j'ai eu des discussion passionnantes au sujet de cartes, de verres de spin, ou d'équations aux dérivées partielles. Merci en particulier à Nikolay d'avoir été dans mon comité de suivi. J'en profite pour remercier une nouvelle fois Jérémie, pour en avoir été l'autre membre.

Je remercie Emmanuel, Laurent, et Grégory de m'avoir fait confiance pour donner les TDs des cours qu'ils enseignaient, et Vanessa, Victor, Mathieu, et Tanguy d'avoir partagé ces TDs avec moi.

Je remercie Gaëtan et Antti, de m'avoir accueilli à Berlin et Genève, et au contact desquels j'ai énormément appris.

Je remercie les communautés MEGA et cartes, que j'ai eu la chance et le plaisir de côtoyer pendant ces trois années.

Et bien sûr, je remercie ma famille, en particulier mes parents, mes grands-parents, mon frère sur qui j'ai toujours pu compter, mes amis, qui m'ont toujours soutenu, et Agathe, sans qui ces trois années n'auraient pas été aussi belles.

[1] Cette thèse a été financé par le projet ERC LDRAM ERC-2019-ADG Project: 884584

[2] Les deux répétitions précédentes sont intentionelles. Indice: je compte aussi les physiciens...

Contents

1 Introduction **11**

1.1 Some questions in random matrix theory . . . 11
1.2 The planar approximation in physics . . . 12
1.3 A few results on the combinatorics of maps . . . 14
1.4 First contribution: Topological expansion of unitary integrals, and maps . . . 15
1.5 Second contribution: Moments of the β-ensemble and maps . . . 17
1.6 Third contribution: Fay formulae of Pfaffian type for hyperelliptic curves . . . 18
1.7 Fourth contribution: Semi-localization in the Generalized Random Graph model . . . 19

2 Preliminaries **23**

2.1 Partitions and permutations . . . 23
2.2 Observables, moments, and cumulants . . . 24
2.2.1 Empirical measure and moments . . . 24
2.2.2 Cumulants, connected and disconnected objects . . . 27
2.2.3 Free probability theory . . . 28
2.3 Symmetries and invariant models . . . 30
2.3.1 Groups of symmetries . . . 30
2.3.2 $\mathbb{U}_\beta(N)$-invariant matrix models . . . 31
2.4 Maps and coverings . . . 33
2.4.1 Maps . . . 33
2.4.2 Coverings and Hurwitz numbers . . . 37
2.5 Computation of moments: direct approach . . . 41
2.5.1 The Wick calculus and maps . . . 41
2.5.2 Moments of the β-ensembles and Jack polynomials . . . 44
2.6 Computation of moments: induction relations . . . 47
2.6.1 The Harer-Zagier recurrence . . . 47
2.6.2 Dyson-Schwinger equations . . . 48
2.7 Geometry and integrable systems . . . 52
2.7.1 A geometric interpretation of the Dyson-Schwinger equations . . . 52
2.7.2 Riemann surfaces and theta functions . . . 52
2.7.3 Fay's formula and some applications . . . 54
2.7.4 Integrable formulae for the β-ensemble . . . 57
2.8 Asymptotics of matrix models . . . 58
2.8.1 Topological expansions . . . 58
2.8.2 Asymptotics of the β-model in the multicut regime . . . 61
2.9 Localization and delocalization of eigenvectors . . . 62
2.9.1 A few remarks on delocalization . . . 62
2.9.2 Adjacency matrices of graphs . . . 62

3 Topological expansion of unitary integrals and maps **65**
3.1 Introduction . . . 65
3.2 Weingarten calculus . . . 68
3.2.1 Moments and cumulants . . . 68
3.2.2 The Weingarten formula . . . 70
3.2.3 Expansion of the Weingarten function . . . 72
3.3 Oriented maps and maps of unitary type . . . 73
3.3.1 Oriented maps . . . 73
3.3.2 Maps of unitary type . . . 74
3.3.3 Permutational model . . . 76
3.3.4 Expression of the moments in terms of maps of unitary type . . . 81
3.3.5 Formal topological expansion . . . 85
3.3.6 Alternated polynomials and Hurwitz numbers . . . 85
3.4 Intermezzo: Voiculescu's Theorem and maps of unitary type . . . 87
3.5 Tutte-like equations . . . 89
3.5.1 How to cut maps . . . 92
3.5.2 Proof of Theorem 3.5.1 . . . 98
3.5.3 Induction relation for the series $\mathcal{M}_{V,l}^{(g),N}$ . . . 101
3.6 The multi-matrix case . . . 102
3.6.1 Weingarten calculus . . . 102
3.6.2 Multicolored maps of unitary type . . . 104
3.6.3 Induction relation . . . 106
3.7 The Dyson-Schwinger equation and the topological expansion . . . 107
3.7.1 Scalar product and parametric norms on $\mathcal{A}_n$ . . . 107
3.7.2 The Dyson-Schwinger equations for the unitary matrices . . . 108
3.7.3 Radius of convergence of the series $\mathcal{M}_{V,l}^{(g),N}$ . . . 108
3.7.4 The topological expansion: proof of Theorem 3.1.4 . . . 110
3.8 Bounds for the sum of maps $\mathcal{M}_{0,l}^{(g),N}$ (Proof of Proposition 3.5.22) . . . 114
3.9 The gradient trick . . . 117
3.9.1 The trick . . . 117
3.9.2 Operator norm estimates . . . 118

4 Moments of the β-ensemble and maps **121**
4.1 Introduction . . . 121
4.2 The moments of the β-ensemble . . . 123
4.2.1 The tridiagonal model . . . 124
4.2.2 Combinatorial interpretation of the χ distribution . . . 124
4.2.3 Moments and Motzkin paths . . . 125
4.2.4 Large N expansion . . . 128
4.3 Maps and labelled hypermaps . . . 130
4.3.1 Maps and permutations . . . 130
4.3.2 Well-labelled hypermaps and suitably labelled maps . . . 132
4.3.3 Encoding the labelled hypermaps . . . 134
4.3.4 Labelling the half-edges in the Bouttier-Fusy-Guitter bijection . . . 137
4.4 Combinatorial description of the cumulants . . . 139
4.4.1 Expression in terms of the distances . . . 139
4.4.2 Analysis of the first two orders . . . 140
4.5 A many-to-one map between suitably labelled maps, and non-orientable maps on $\mathbb{RP}^2$ . . . 142
4.5.1 The orientation double covering . . . 142
4.5.2 Combinatorial description of non-orientable maps . . . 143

4.5.3 Maps on the orientation covering . . . 144
4.5.4 Cutting and gluing suitably labelled maps . . . 147
4.5.5 Choosing a path in a map . . . 152
4.5.6 The mapping for maps on $\mathbb{RP}^2$ . . . 159
4.6 The limit $\beta \to \infty$ and the roots of Hermite polynomials . . . 159

5 Fay identities of Pfaffian type for Hyperelliptic curves 161
5.1 Introduction . . . 161
5.2 β-ensembles and their properties . . . 162
5.2.1 The unconstrained model . . . 162
5.2.2 The model with fixed filling fractions . . . 163
5.2.3 Equilibrium measures and their Stieltjes transform . . . 163
5.2.4 Determinantal and pfaffian formulae . . . 166
5.3 Geometry of the spectral curves . . . 168
5.3.1 Theta functions . . . 168
5.3.2 Geometry of Riemann surfaces . . . 168
5.3.3 The spectral curve . . . 171
5.3.4 Characterisation of spectral curves of β-ensembles . . . 174
5.3.5 Deformations of the curve . . . 175
5.4 Asymptotics of the partition function and the kernels . . . 177
5.4.1 Expansion of the partition function and generalised central limit theorem . . . 177
5.4.2 Three explicit formulae . . . 177
5.4.3 Kernel asymptotics: intermediate computations . . . 179
5.5 Derivation of the theta identities . . . 183
5.5.1 The $\beta = 2$ formula . . . 183
5.5.2 The $\beta = 1$ formula . . . 188
5.5.3 The $\beta = 4$ formula . . . 191
5.5.4 Formula for the multi-cut equilibrium energy (Proof of Proposition 5.4.3) . . . 191
5.6 Variation of the entropy with respect to filling fractions (Proof of Proposition 5.4.4) . . . 192

6 Semi-localization in the Generalized Random Graph model 195
6.1 Introduction . . . 195
6.1.1 Semi-localization result . . . 197
6.1.2 The degrees and the weights . . . 197
6.2 Pruning the graph . . . 198
6.2.1 The set of vertices of high degree . . . 198
6.2.2 Removing the cycles . . . 199
6.2.3 Removing the down-up paths . . . 202
6.2.4 The pruning procedure . . . 204
6.2.5 Estimate of $\|A - A^{\mathrm{p}}\|$ . . . 205
6.3 Finding the eigenvectors . . . 207
6.4 The spectral gap and the semi-localization phenomenon . . . 215
6.4.1 The block-diagonal approximation . . . 215
6.4.2 Bounds on $\overline{\Pi}A^{\mathrm{p}}\overline{\Pi}$ . . . 217
6.4.3 Semi-localization . . . 218
6.5 The biggest eigenvalues and the localization phenomenon . . . 220
6.6 Bounds for the proof of Proposition 6.4.1 . . . 222
6.7 Estimation of the size of $\mathcal{W}_{\lambda,\eta}$ (Proof of Proposition 6.1.8) . . . 223

7 Future research **227**
7.1 Topological expansion of unitary integrals, and beyond 227
7.2 β-ensembles and maps 227
7.3 Other Fay-like identites 228
7.4 Different regimes for the localization in the GRG model 228
7.5 A new point of view on Dyson-Schwinger equations 228
7.6 Large deviations for the empirical measure of sub-Gaussian Wigner matrices 229

8 Appendices **231**
8.1 Derivation of the Dyson-Schwinger equations 231
8.2 Coverings of higher genus surfaces 232

Chapter 1

Introduction

We start by motivating our main contributions through an overview of the main themes of this Thesis. The mathematical notions we discuss informally in this introduction will be defined rigorously in Chapter 2.

1.1 Some questions in random matrix theory

Random Matrix Theory originated from many sources. One could argue, like Diaconis and Forrester [DF16], that it started with the work of Hurwitz [Hur97] on the Haar measure of classical groups. Hurwitz gave a parametrization of the group $SO(N)$ in terms of Euler angles, which allowed him to express the Haar measure in a simple way. However, this point of view is centered on the algebraic part of the theory. The origin of Random Matrix Theory as a part of probability theory may rather be traced back to Wishart's work in statistics [Wis28]. Wishart was interested in $p \times N$ rectangular matrices X, made of p families of N samples. His goal was to estimate the empirical variances and covariances between these p families. He thus introduced rectangular random matrices with independent entries as a main object of study. The topic entered mathematical physics with the work of Wigner [Wig55; Wig58], related to nuclear physics. In particular, he introduced in [Wig58] the random Hermitian matrices with identically and independently distributed entries (up to Hermiticity) that now bear his name. Wigner matrices appeared as a model Hamiltonian for a heavy nucleus, whose eigenvalues could be computed (see also the review [Wig67]). He showed that the eigenvalues of such matrices are distributed in the large dimension limit as the semicircle distribution, ubiquitous in random matrix theory.

Wigner's interest was to understand the spectrum of some random matrices, that corresponds to energy levels of heavy nuclei. Soon after this work, Anderson [And58] proposed a random matrix model in condensed matter physics. His goal was to describe quantum diffusion in a random medium: the behaviour of the eigenvectors rather than the eigenvalues was the question of interest. If an eigenvector is (roughly) supported on a small number of components, we say that it is *localized*, and *delocalized* otherwise. The random Anderson Hamiltonian is made of a diffusion part on a lattice (that may or may not be random) and a random potential part, effectively a random diagonal in the matrix. Despite the apparent simplicity of the model, many questions related to Anderson localization remain open. A way to probe this physical question is to consider graph models. The adjacency matrix of a random graph can then be thought of as being an analogue of the random diffusive part of a random Hamiltonian.

We stressed that Wishart and Wigner matrices are assumed to have independent entries. Building on works of Wigner, Dyson [Dys62a; Dys62b] argued that while independence is a convenient mathematical assumption, it does not describes the relevant physical invariances in most situations. He argued that the truly physical assumption was that of invariance of the measure under the action of a group of symmetries. The group of invariance we shall be interested in, as well as the corresponding matrix models, will be discussed in Section 2.3.

Following Wigner, we will approach the study of the spectrum of random matrices by studying the *moments* of the empirical distribution of the eigenvalues of our random matrices. By moments of a random matrix X, we mean quantities of the form

$$\mathbb{E}\left[\operatorname{Tr} X^k\right] \qquad \text{for } k \geq 0.$$

The joint moments are then

$$\mathbb{E}\left[\operatorname{Tr} X^{k_1} \cdots \operatorname{Tr} X^{k_l}\right] \qquad \text{for } k_1, \ldots, k_l \geq 1.$$

We will discuss moments in Section 2.2. A surprising fact is that in many cases, such quantities can be expressed in terms of enumeration of *maps*. Informally, maps are graphs drawn on surfaces, or alternatively, discretized surfaces, made of several polygons glued together along their edges. They will be defined formally in Section 2.4.1. This point of view was pioneered by physicists such as 't Hooft [tH74] in the context of quantum field theory. We start be giving a short account of how the description of the moments of the GUE in terms of maps appeared in physics.

1.2 The planar approximation in physics

In quantum field theory, one wishes to compute complicated integrals over spaces of classical fields. Those classical fields are functions $X \to Y$ between a space X, a model of the universe, and a target space Y. Defining the integration procedure is often a difficult problem which may in some cases be tackled by an appropriate discretization scheme. In many cases, we can define the integration when the fields considered are not interacting together. Roughly speaking, this means we are integrating over the space of classical fields under a generalization of the Gaussian measure. This case is usually called the *free field*. Once this is defined, many computations can be done using perturbation theory. In this setup, to make the computation tractable, the interaction terms between the fields are assumed to be a perturbation of the free field. A parameter scaling the strength of the interaction is introduced and assumed to be small. It allows to express the integral under study as a series in the strength parameter of the interaction. The coefficients of the series are described using a graphical method: to each contribution to the series corresponds a Feynman diagram, a graph with different types of vertices and edges, all of which admit a physical interpretation. Each Feynman diagram is weighted by an integral encoded by the diagram.

An important step toward the introduction of topological expansion is the work of 't Hooft on the strong interaction [tH74]. He proposed a model in which he considered N types of particles in interaction, with a coupling parameter g[1]. In the limit $N \to \infty$ and $g \to 0$ with $\lambda = g^2 N$ kept constant, he remarked that among the Feynman diagrams for his model, only the planar ones gave a contribution. The contribution of the other diagrams decay with a power of N depending on their genus. This highlighted that models with a large number N of internal degrees of freedom (in this case, a large number of different type of particles or colors) simplify considerably when $N \to \infty$.

A few years later, Brézin, Itzykson, Parisi, and Zuber [Bré+78] continued the study of this simplification in the planar limit. They considered a toy quantum field theory in which the model universe is a point and the target space of the classical field is the space of Hermitian $N \times N$ matrices $\mathcal{H}_2(N)$. In fact, they considered (up to a different scaling) the following integral:

$$Z_{2,V}^N = \int_{\mathcal{H}_2(N)} \mathrm{e}^{-N \operatorname{Tr} V(X) - \frac{N}{2} \operatorname{Tr} X^2} \mathrm{d}X,$$

where $V = \sum_{k \geq 1} \frac{t_k}{k} x^k$ is a polynomial and $\mathrm{d}X$ denotes the Lebesgue measure on the space $\mathcal{H}_2(N)$ of $N \times N$ Hermitian random matrices. This integral can be seen as a perturbation of a free field: up

[1] In quantum chromodynamics, the particles studied are quarks, which may have different colors. Usually, the number of colors is chosen to be 3 (corresponding to the gauge group $SU(3)$). 't Hooft considered a model in which the number of colors is N, taken to be large (this corresponds to the gauge group $SU(N)$).

to a normalization factor $Z^N_{2,0}$, the weight $\exp\left(-\frac{N}{2}\operatorname{Tr}\left(X^2\right)\right)\mathrm{d}X$ is the Gaussian measure on $\mathcal{H}_2(N)$, called the Gaussian Unitary Ensemble (GUE), and $\exp(-NV(X))$ is the perturbation, provided the coefficients (t_i) are small.

For the sake of simplicity, let us consider the case $V(X) = gX^4$, the quartic potential. This integral can be studied in two ways in the limit $N \to \infty$:

- the quantity $\frac{1}{N^2}\ln\frac{Z^N_{2,gx^4}}{Z^N_{2,0}}$ is the free energy of a system of N interacting particles (the eigenvalues of the random matrix X) which can be studied through the steepest descent method,
- using Gaussian calculus as in the work of 't Hooft, $\lim_{N\to\infty}\frac{1}{N^2}\ln\frac{Z^N_{2,gx^4}}{Z^N_{2,0}}$ can be shown to be (at least formally) the generating function of quadrivalent planar maps.

We will discuss the second point in detail in Section 2.5.1, and for now give an informal discussion. The computation of the moments under the appropriate Gaussian measure, the GUE, are carried out using the Isserlis-Wick formula. In essence, this formula relates the expectation of any product of centered Gaussian variables, to sum of products of covariances. It allows to show that the moments of the GUE can be expressed as sums over maps. Working at the level of formal power series, the results of Brézin et al. directly imply:

$$\lim_{N\to\infty}\frac{1}{N^2}\ln\frac{Z^N_{2,gx^4}}{Z^N_{2,0}} = \sum_{n\geq 1}\frac{g^n}{n}\#\left\{\mathfrak{m}: \begin{array}{c}\mathfrak{m}\text{ is an orientable rooted}\\ \text{planar map}\\ \text{with vertices of degree } 4\end{array}\right\}.$$

These two different ways to tackle the computation of the limit of the free energy make it possible to solve the combinatorial problem of counting some planar maps with random matrix tools. This method was developed in details in [BIZ80] for the case of the quartic potential. In Section 2.5.1, we shall be interested in the second technique, which links matrix integrals to combinatorial problems. Note that the random matrix considered here is invariant under conjugation by a $N \times N$ unitary matrix. This is, as in 't Hooft's model, a model with a large number of degrees of freedom.

We just saw that a one-matrix model can be seen as a quantum field theory with a universe reduced to one point (a dimension zero universe). This remark generalizes to universes of dimension zero which are a finite number m of points. A multi-matrix model with m matrices can thus be seen as a quantum field theory over such a universe.

We remark that during the developments we recounted, the Feynman diagrams are considered in two different ways: as tools for computing complicated integrals in 't Hooft's model (combinatorics is used to obtain analytic result), and as a set of combinatorial objects that we can study in their own right in Brézin et al.'s work (analytic methods are used to obtain a combinatorial result).

This point of view was used many times in the physics literature, in particular in connection with 2D quantum gravity. An important technical question of 2D quantum gravity is to be able to integrate over random surfaces, as famously emphasized by Polyakov [Pol81]. A point of view (see the review [DGZ95] and references inside) was to consider a finite sum over discretized surfaces, that is, maps. The matrix model techniques of Brézin et al. thus appeared as an analytically tractable way to study those discretized surfaces and investigate their continuum limit.

In statistical physics, this approach was used in particular by Kazakov [Kaz86], and Kazakov and Boulatov [BK87], to study the Ising model on a random quadrivalent planar lattice. It corresponds to considering the 2-matrix model with potential

$$V(X,Y) = gX^4 + gY^4 - cXY.$$

As we shall explain later, in Remark 2.5.3, the parameter g controls the number of quadrivalent vertices in the lattice, and the parameter c the number of frustrated edges of the model. This matrix model was studied first by Itzykson and Zuber in [IZ80] and soon after by Mehta [Meh81].

1.3 A few results on the combinatorics of maps

The analytic approach to the combinatorics of maps of Brézin et al. was preceded by the foundational work of Tutte on planar maps [Tut62a; Tut62b; Tut62c; Tut63]. One remarkable result shown by Tutte was an exact formula for the number of planar maps for a given number of edges and prescribed degrees of vertices, known as the "slicing formula" [Tut62c]. He proposed an approach centered on generating series to study planar maps. He introduced a surgery on planar maps that allowed him to obtain an equation on generating series of planar maps, now known as Tutte's equation [Tut68]. This will be explained in Section 2.4.1. The decomposition of Tutte was extended to any genus by Walsh and Lehman [WL72a; WL72b]. Using such methods, Bender and Canfield [BC94] gave formulae for counting the number of planar maps with vertex degrees lying in a particular set. Hence, by the method of Tutte, the results of Brézin et al. could be obtained by purely combinatorial techniques.

The combinatorial approaches are more satisfying than the matrix integral approach, which hide away the details of the combinatorics of planar maps. In particular, bijective methods – methods based on bijections between families of combinatorial objects – allow a very complete understanding of enumerative formulae.

Pioneering work on these bijective arguments was done by Cori [Cor75], who gave a bijective proof of the "slicing formula" of Tutte. Cori and Vauquelin [CV81] then introduced their celebrated bijection, relating rooted planar maps and a family of trees with labelled vertices. A similar bijection for the case of hypermaps (or maps with bi-colored vertices) was proposed by Arques [Arq86]. Building on these developments, a number of results of Tutte were re-proved in a bijective way, and extended, by Schaeffer [Sch98]. The construction of Schaeffer gives naturally a bijection between labelled trees and planar quadrangulations, i.e. planar maps with vertices of degree four. However, an elementary bijection due to Tutte relates planar maps with n edges to quadrangulations with n vertices, thus showing that in particular, treating the case of quadrangulations suffices for the enumeration of maps with a given number of edges. Bouttier, di Francesco, and Guitter [BDG04] subsequently provided an analogue to the bijections of Cori and Vauquelin, and of Schaeffer, between bi-colored planar map with prescribed vertex degrees, and a family of labelled trees called *mobiles*. Most of the bijections up to now were bijection between planar maps and labelled trees. The bijection of Cori-Vauquelin-Schaeffer was extended to a bijection between orientable maps of higher genus and one-face labelled maps of higher genus by Chapuy, Marcus, and Schaeffer [CMS09], and then, to the case of non-orientable maps by Chapuy and Dołęga [CD17]. The Bouttier-di Francesco-Guitter bijection was extended to orientable maps of higher genus by Chapuy [Cha09], and to non-orientable maps by Bettinelli [Bet22].

By relying on these techniques, the results of the physicists we mentioned in Section 1.2 were recovered, and improved, by purely combinatorial methods. For instance, Bouttier, di Francesco, and Guitter [BDG02] obtained the results of Brézin et al. with a purely bijective method. Similarly, the Ising model studied analytically by Kazakov was studied combinatorially by Bousquet-Melou and Schaeffer [BS03]. They provided a very general technique to express the generating series of the Ising model on a random map, with a specified vertex degree distribution.

Interestingly, the bijections relating planar maps to labelled trees discussed so far also give insight in the geometric properties of the planar maps, such as the graph distance to a marked vertex, as studied in [CS02; BDG03a; BDG03b]. The study of graph distances in planar maps started with the predictions of Ambjørn and Watabiki [AW95] in physics. The two point function, which describes the distribution of distances between two random points in a planar map, was computed for several families of maps by Bouttier, di Francesco, and Guitter [BDG03a], and di Francesco [DiF05]. A general approach for general maps with bounded vertex degrees was given by Bouttier and Guitter [BG12]. The case of unbounded vertex degrees was studied by Bouttier, Fusy, and Guitter [BFG14]. In the latter article, they gave a further generalization of the maps-to-mobile bijection that will be used in Chapter 4.

Our first two results are at the crossroads of the matrix model approach and the combinatorial approach: one one hand, they are random matrix results which offer insight on the enumeration of some maps, and on the other hand, they rely on combinatorial techniques such as Tutte's decomposition or bijective methods.

1.4 First contribution: Topological expansion of unitary integrals, and maps

The result of Brézin, Itzykson, Parisi, and Zuber was extended to a full asymptotic topological expansion by Ercolani and McLaughlin [EM03] in the perturbative regime, i.e. when V has small enough coefficients. More precisely, they showed that in that regime, for all $g \geq 0$,

$$\frac{1}{N^2} \ln \frac{Z_{2,V}^N}{Z_{2,0}^N} = \mathcal{M}_0(V) + \frac{1}{N^2}\mathcal{M}_1(V) + \cdots + \frac{1}{N^{2g}}\mathcal{M}_g(V) + \mathcal{O}\left(\frac{1}{N^{2g+2}}\right), \tag{1.1}$$

with $\mathcal{M}_h(V)$ a generating series of maps of genus h, i.e. graphs drawn on a torus with h holes. This expansion is called topological as the genus of the maps, a topological invariant, governs the order of the terms. This expansion was shown to hold in the multi-matrix case by Guionnet and Maurel-Segala [GM06; GM07], and Maurel-Segala [Mau06]: models obtained by a perturbation of the law of $m \geq 1$ independent GUE matrices were shown to display the same kind of expansion as in (1.1). In the approach of Guionnet and Maurel-Segala, the key tool to upgrade the result of Brézin et al., that holds at the level of formal power series, to an asymptotic expansion, is the Dyson-Schwinger family of equations. The Dyson-Schwinger equations relate different joint moments of our random matrices together. They are obtained by integration by part. For instance, the first Dyson-Schwinger equation in the one-matrix case is:

$$\sum_{i+j=n} \mathbb{E}\left[\operatorname{Tr} X^i \operatorname{Tr} X^j\right] = \mathbb{E}\left[\operatorname{Tr}\left(V'(X) + X\right) X^{n+1}\right].$$

The Dyson-Schwinger equations will be discussed in Section 2.6.2. Two important facts about the Dyson-Schwinger equations are relevant for us:

- in the perturbative regime and in the large N limit, the Dyson-Schwinger equations admit a unique solution;
- generating series of maps are solutions to the Dyson-Schwinger equations.

The latter point is due to Tutte [Tut68]: it turns out that Tutte's equations alluded to earlier happen to coincide with Dyson-Schwinger equation for a perturbation of the GUE. The two above points were successfully used by Guionnet and Maurel-Segala to show the existence of the topological expansion in the multi-matrix case, and express it in terms of maps.

The main result of Chapter 3 (based on [Buc24]), Theorem 1.4.1 below, is the analogue of the result of Guionnet and Maurel-Segala, in the case of integrals over the unitary group $\mathbb{U}(N)$. In this case, we consider integrals against perturbations of the Haar measure, of the form

$$\mu_{\mathbb{U},V}^N = \frac{1}{Z_{\mathbb{U},V}^N} \exp(N \operatorname{Tr} V(U, U^*, A_1, \ldots, A_p)) \mathrm{d}U,$$

with $Z_{\mathbb{U},V}^N$ a normalization constant, $\mathrm{d}U$ the Haar measure, and V a non-commutative polynomial in the matrices U and U^*, and in $p \geq 0$ deterministic square $N \times N$ matrices $A_1, \ldots, A_p$, i.e. linear combination of products of matrices $U, U^*, A_1, \ldots, A_p$. This question was studied by Collins, Guionnet, and Maurel-Segala [CGM09], who gave the leading order of integrals against $\mu_{\mathbb{U},V}^N$ in terms of a family of planar maps. A full asymptotic expansion was then given by Guionnet and Novak [GN15]. This expansion, however, was not topological: the coefficients of the expansion were not identified with series of maps. We were able to prove results of the following type.

Theorem 1.4.1. *Assume that for all $N \geq 1$, $\mathrm{Tr}(V)$ is real for all $U \in \mathbb{U}(N)$ and that*

$$\sup_{N\geq 1}\sup_{1\leq i\leq p}\|A_i\| < \infty.$$

Then, there exists $\epsilon > 0$ such that if the coefficients of V are bounded by ϵ, then for all $g \geq 0$, and non-commutative polynomial P, we have the asymptotic expansion as $N \to \infty$

$$\frac{1}{N}\mathbb{E}\left[\mathrm{Tr}\, P(U, U^*, A_1, \ldots, A_p)\right] = \sum_{h=0}^{g} \frac{1}{N^{2h}} \mathcal{M}_{V,1}^{(h),N}(P) + \mathcal{O}\big(N^{-2g-2}\big),$$

where $\mathcal{M}_{V,1}^{(h),N}(P)$ are generating series of maps of unitary type of genus h whose structure depends on V and P.

In the above result, the dependence in N of $\mathcal{M}_{V,1}^{(h),N}(P)$ only comes from the dependence in N of the matrices $(A_k)_{1\leq k\leq p}$. The actual result showed in Chapter 3, Theorem 3.1.4, holds in the multi-matrix case, and for all cumulants (introduced in Section 2.2.2).

In both the result of Collins, Guionnet and Maurel-Segala [CGM09], and in the result of Guionnet and Novak [GN15], the main tool was the Dyson-Schwinger equations for the deformation of the Haar measure, similar to the ones for Hermitian matrices. To derive Theorem 1.4.1, we adopt a different point of view, and stick more closely to the way the multi-matrix expansion was obtained in the Hermitian case: we first express moments of the Haar measure in terms of maps to obtain a formal expansion, and then upgrade the formal expansion to an asymptotic expansion using the Dyson-Schwinger equations. To express the moments under the Haar measure, we make use of the Weingarten calculus, which replaces the Gaussian calculus in our setup. The Weingarten calculus for the unitary group was developed by Samuel [Sam80], based on previous work by Weingarten [Wei78]. It was subsequently rediscovered and expanded upon by Collins [Col03], and Collins and Śniady [CŚ06]. We will use in particular expressions derived by Novak [Nov10]. Using Weingarten calculus, we express joint moments under the Haar measure in terms of a family of maps, which we call *maps of unitary type*. We show that the unitary Dyson-Schwinger equations can be seen as an analogue of Tutte's equation for the maps of unitary type. This allowed us to prove a results of the form of Theorem 1.4.1. An interesting point is that beyond showing that the expansion is topological, our result improve the expansion of Guionnet and Novak, as the bound on the coefficient of the potential V does not depend on the number of terms in the expansion.

In particular cases, the combinatorics of maps of unitary type reduces to the combinatorics of monotone Hurwitz numbers. These numbers appeared as the main combinatorial object in the asymptotic expansion of the Harish-Chandra-Itzykson-Zuber (HCIZ) integral derived by Guay-Paquet, Goulden, and Novak [GGN14]. This integral is related to the Ising model mentioned in the previous section. It is defined by

$$\int_{\mathbb{U}_2(N)} \mathrm{e}^{zN\,\mathrm{Tr}(AUBU^*)}\mathrm{d}U.$$

It was studied by Harish-Chandra [Har57], and independently by Itzkyson and Zuber [IZ80]. Furthermore, the study of unitary integrals is a way to obtain results in free probability theory, which in particular describes moments of random matrices in the large N limit. Free probability uses combinatorial objects in a way that will often be paralleled by the the point of view centered on maps, which we adopt in this thesis. In particular, monotone Hurwitz numbers play a key role in the functional relations of all order freeness, given by Borot, Charbonnier, Garcia-Failde, Leid, and Shadrin [Bor+21]. An expansion such as the one of Theorem 1.4.1 provides insights on the combinatorics of monotone Hurwitz numbers, and on the combinatorics of higher order free probability.

Beyond the interest in combinatorics, knowing the sub-leading order of the moments allows to show a central limit theorem: the fluctuations of the normalized trace of any polynomial in the eigenvalues can be shown to be Gaussian, see for instance [GN15, Corollary 4]. The higher orders in the expansion give additional information on the fluctuations of such traces.

1.5 Second contribution: Moments of the β-ensemble and maps

The preceding section discussed one way to extend the results of Brézin et al., by going from the Gaussian matrix of the GUE, to unitary matrices. In Chapter 4, we rather consider moments under the measure $\nu^N_{\beta,0}$ of the β-ensemble defined by

$$\mathrm{d}\nu^N_{\beta,0}(\lambda_1,\dots,\lambda_N) = \frac{1}{\mathcal{Z}^N_{\beta,0}} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \, \mathrm{e}^{-\frac{\beta N}{4}\sum_i \lambda_i^2} \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N.$$

When $\beta = 2$, this is the distribution of the N eigenvalues $\lambda_1,\dots,\lambda_N$ of a GUE matrix. Two other remarkable values are $\beta = 1$ and $\beta = 4$, corresponding respectively to the eigenvalues of a Gaussian real symmetric matrix – in the Gaussian Orthogonal Ensemble (GOE) – and to the eigenvalues of a Gaussian quaternionic self-adjoint matrix – in the Gaussian Symplectic Ensemble (GSE). At the level of the eigenvalues, the moments and joint moments are the expectation of the power sums of the eigenvalues, respectively

$$\mathbb{E}\left[p_{(k)}(\lambda_1,\dots,\lambda_N)\right] = \sum_{i=1}^N \lambda_i^k \quad \text{for } k \geq 1,$$

and

$$\mathbb{E}\left[\prod_{i=1}^l p_{(k_i)}(\lambda_1,\dots,\lambda_N)\right] = \sum_{i=1}^N \lambda_i^k \quad \text{for } k_1,\dots,k_l \geq 1.$$

Out of the integrable cases $\beta \in \{1,2,4\}$, the method of Brézin et al. based on Gaussian calculus is not applicable, and the moments may not be directly expressed in terms of maps in the same way as before. However, such an expression was provided using techniques from algebraic combinatorics. Goulden and Jackson [GJ97] noticed that while the moments of the GUE are related to the combinatorics of orientable maps, i.e. graph drawn on orientable surfaces, the moments of the GOE and GSE are related to the combinatorics of all maps, possibly on non-orientable surfaces. They showed that the generating function of orientable maps M_0 and of all maps M_1 could be expressed in terms of Schur or zonal polynomials respectively. Schur and zonal polynomial are two families of symmetric polynomial related to integration over the unitary and orthogonal group. A third family of symmetric polynomial, the Jack polynomials, depends on a parameter b and interpolates between the Schur $b = 0$ and zonal $b = 1$ polynomials. Replacing the Schur or zonal polynomials by Jack polynomials in M_0 and M_1, Goulden and Jackson defined a new generating function M_b. They conjectured [GJ96] that this function is a generating function of orientable and non-orientable maps, each map $\mathfrak{m}$ weighted by $b^{\vartheta(\mathfrak{m})}$, with $\vartheta(\mathfrak{m})$ a measure of how non-orientable $\mathfrak{m}$ is. Quickly after that, Goulden, Harer, and Jackson [GHJ01] showed that a particularization of M_b with $b = \frac{2}{\beta} - 1$ was related to the free energy of the β-ensemble

$$\frac{1}{N^2} \ln \nu^N_{\beta,0}\left[\mathrm{e}^{\frac{\beta}{2}\sum_k \frac{t_k}{k}\sum_i \lambda_i^k}\right].$$

Hence, provided a particular case of the Goulden-Jackson conjecture is true, the moments of the β-ensemble can be expanded as sums over maps, each map $\mathfrak{m}$ weighted by $N^{-\chi(\mathfrak{m})}b^{\vartheta(\mathfrak{m})}$, with $\chi(\mathfrak{m})$ the Euler characteristic of the map $\mathfrak{m}$. This particular version of the Goulden-Jackson conjecture was shown to be true by LaCroix [LaC09], thus giving a way to express combinatorially the moments of the β-ensemble. Beyond the moments of the β-ensemble, the b-conjecture of Goulden and Jackson is still a partially open problem in algebraic combinatorics. In particular, Chapuy and Dołęga [CD22] gave very general results on a generalization of the conjecture of Goulden and Jackson, using ideas from the theory of integrable systems.

The goal of Chapter 4 is to propose a more direct approach to compute the moments of the β-ensemble. We stressed that in the case $\beta \notin \{1,2,4\}$, the Gaussian calculus of Isserlis-Wick is not available. However, a celebrated result of Dumitriu and Edelman is that the β-ensemble is the distribution of eigenvalues of a tridiagonal real symmetric matrix whose entries are either distributed as Gaussian or χ random variables. Abdesselam, Anderson, and Miller [AAM14] used that fact to compute the leading order of the joint moments of the GUE (β-ensemble at $\beta = 2$) in terms of the mobiles introduced by Bouttier, di Francesco, and Guitter [BDG04]. We simplify and extend the approach of Abdesselam et al. and compute an exact expansion of the joint moments of the β-ensemble for all $\beta > 0$. The key combinatorial object are orientable suitably labelled maps, a class of maps with labelled vertices, which are in bijection with well-labelled hypermaps, which generalize mobiles. This bijection was obtained by Bouttier, Fusy, and Guitter [BFG14]. A simplified version of our result, whose general statement is Theorem 4.1.2, is the following.

Theorem 1.5.1. *Let $n \geq 2$ be even, and $\theta = (1\,2\,\ldots\,n)$ a cyclic permutation of order n. We have the following expansion for the moments of the β-ensemble:*

$$\frac{\nu^N_{\beta,0}\left[p_{(n)}(\lambda_1,\ldots,\lambda_N)\right]}{N} = \sum_{v=0}^{n/2} \frac{1}{N^v} \sum_{u+q+r=v} \left(\frac{2}{\beta}\right)^u \frac{(-1)^q B_r}{n/2-1-v} \binom{r+n/2-v}{r} \langle e_q \rangle_{\theta,u}\,, \qquad (1.2)$$

where $\langle\cdot\rangle_{\theta,p}$ denotes a sum over suitably labelled maps whose structure depend on θ and p, introduced in Section 4.4.1. The sequence $(B_r)_{r\geq 0} = (1, -1/2, 1/6, \ldots)$ is the sequence of Bernoulli numbers, defined inductively by

$$\sum_{k=0}^{n} \binom{n+1}{k} (-1)^k B_k = \delta_{n,0} \text{ for all } n \geq 0.$$

A similar result holds for all joint moments of the β-ensemble. In fact we will prove the result in terms of cumulants, defined in Section 2.2.2.

Interestingly, this expansion is quite different from the one obtained by LaCroix, which features only positive terms, and a sum over all maps, orientable and non-orientable. We were able to show bijectively that the first two orders in N^{-1} of both expansions coincide. This is done by using a novel many-to-one mapping relating some suitably labelled maps to maps on the projective plane $\mathbb{RP}^2$.

1.6 Third contribution: Fay formulae of Pfaffian type for hyperelliptic curves

In Chapter 5, based on a joint work with Gaëtan Borot [BB24], we derive geometric identities with methods of random matrix theory. The first Dyson-Schwinger equation mentioned in Section 1.4 can be seen in a geometric way, as an equation defining a complex curve, the *spectral curve*. This geometric point of view appeared in mirror symmetry [DV02], and was subsequently used by Eynard [Eyn05], and then Eynard and Orantin [EO08], to recast the Dyson-Schwinger equations in the language of geometry of Riemann surfaces. In this form, dubbed *topological recursion*, Dyson-Schwinger-like equations could be shown to govern the behavior of many numbers of combinatorial or geometric interest. If we remain in the world of random matrix theory, this geometric point of view centered around the spectral curve allows us to reinterpret observables of random matrix theory in terms of geometric objects related to the spectral curve.

Besides, it was shown by Borot and Guionnet [BG24], that the large N asymptotics of the β-ensemble could be computed in terms of a Riemann theta function, whose parameters depend on the spectral curve. Riemann theta functions are functions of two parameters $(z, \boldsymbol{\tau}) \mapsto \theta(z \mid \boldsymbol{\tau})$ with $z \in \mathbb{C}^g$ and $\boldsymbol{\tau}$ a matrix of size $g \times g$, which appear frequently in the geometry of Riemann surfaces. Informally, those are elliptically deformed trigonometric functions with convenient quasi-periodicity properties.

In Chapter 5, we consider exact formulae of Borodin and Strahov, which relate different averages of characteristic polynomials of our random matrices together. If we use the two ingredients we mentioned: take the large N asymptotics of theses formulae, and re-express the limit identities in terms of geometric quantities of the spectral curves, we obtain new identities between theta functions. For instance, when

$\beta = 1$, we obtain the following result. The geometric notions used to state this theorem will be defined in Section 2.7.

Theorem 1.6.1. *Consider a hyperelliptic curve $\hat{C}$. Let z_1, z_1', z_2, z_2' be in the universal covering of $\hat{C}$, and let $x \in \mathbb{C}^g$. We have*

$$\begin{aligned} &c_1(z_1, z_1', z_2, z_2')\theta\big(x + \boldsymbol{u}(z_1) - \boldsymbol{u}(z_1') + \boldsymbol{u}(z_2) - \boldsymbol{u}(z_2')\big|\tfrac{\boldsymbol{\tau}}{2}\big)\theta\big(x\big|\tfrac{\boldsymbol{\tau}}{2}\big) \\ &= c_2(z_1, z_1', z_2, z_2')\theta\big(x + \boldsymbol{u}(z_1) - \boldsymbol{u}(z_2')\big|\tfrac{\boldsymbol{\tau}}{2}\big)\theta\big(x + \boldsymbol{u}(z_2) - \boldsymbol{u}(z_1')\big|\tfrac{\boldsymbol{\tau}}{2}\big) \\ &\quad + c_3(z_1, z_1', z_2, z_2')\theta\big(x + \boldsymbol{u}(z_1) - \boldsymbol{u}(z_1')\big|\tfrac{\boldsymbol{\tau}}{2}\big)\theta\big(x + \boldsymbol{u}(z_2) - \boldsymbol{u}(z_2')\big|\tfrac{\boldsymbol{\tau}}{2}\big) \\ &\quad + c_4(z_1, z_1', z_2, z_2')\theta\big(x + \boldsymbol{u}(z_1) + \boldsymbol{u}(z_2) - \boldsymbol{u}(\infty_-)\big|\tfrac{\boldsymbol{\tau}}{2}\big)\theta\big(x - \boldsymbol{u}(z_1') - \boldsymbol{u}(z_2') + \boldsymbol{u}(\infty_-)\big|\tfrac{\boldsymbol{\tau}}{2}\big), \end{aligned}$$

where $c_i, i \in \{1, 2, 3, 4\}$ are explicit meromorphic functions of the four points $z_1, z_1', z_2, z_{2'}$, $\boldsymbol{\tau}$ is a choice of period matrix for $\hat{C}$, a matrix defined in terms of $\hat{C}$, and $\boldsymbol{u}$ is the Abel map of $\hat{C}$, a mapping from the universal covering of $\hat{C}$ to $\mathbb{C}^g$ well-defined up to a choice of base point in $\hat{C}$. The two points ∞_+ and ∞_- are points at infinity, related by the hyperelliptic involution.

Such an identity is reminiscent of Fay's identity, an identity between theta functions valid for any Riemann surface, which we discuss in Section 2.7.3. It has applications in geometry and in the theory of integrable systems. Note that our result is valid only for hyperelliptic curves, a special family of Riemann surfaces. This limitation comes from the Dyson-Schwinger equations: seen in a geometric light they are equations of hyperelliptic curves only. More complicated models than the β-ensemble, for instance involving several interacting groups of particles, would be needed to treat a greater class of curves.

1.7 Fourth contribution: Semi-localization in the Generalized Random Graph model

In Chapter 6, based on a joint work with Antti Knowles, we study the eigenvectors – rather than the spectrum – of random matrices, and use quite different tools from those presented in Sections 1.4, 1.5, and 1.6.

We are interested in the notion of localization and delocalization of eigenvectors. Indeed, for $N \in \mathbf{N}^*$, a normalized eigenvector of a $N \times N$ matrix can be seen as a function $[N] \to \mathbb{C}$ supported on N sites. Informally, the vector is localized if it has a big absolute value on a few sites. On the contrary, it is delocalized if it has roughly the same absolute value on all sites. The relevant mathematically precise definition differs depending on the application: the localized-delocalized behavior is a continuum rather than a dichotomy. The criterion we can use for now is that a normalized vector $\boldsymbol{v}$ in $\mathbb{C}^N$ is delocalized if $\|\boldsymbol{v}\|_\infty$ is close to $1/N$, and is localized if $\|\boldsymbol{v}\|_\infty$ is close to 1.

The question of whether eigenvectors of a random matrix are localized or not appeared first in the seminal work of Anderson [And58] in condensed matter physics. Anderson was interested in the quantum diffusion in lattices subjected to a noisy potential. He predicted that depending on the strength of the noise[2], the system undergoes a transition from the "metallic phase" (low noise, delocalization of eigenfunctions) to the "insulating phase" (higher noise, localization of eigenfunctions). A comprehensive description of this phenomenon from the physical point of view can be found in the review [EM08]. A landmark result in the mathematical community was obtained by Fröhlich and Spencer [FS83], who showed that in the regime of strong disorder the eigenvectors are localized. Aizenman and Molchanov [AM93] subsequently obtained a similar result for a variety of models generalizing the Anderson model.

From the point of view of random matrix theory, the Anderson model is a random matrix X^N written

$$X^N = A^N + V^N,$$

[2] Anderson considered a three-dimensional lattice, but the localization phenomenon depends of the dimension. In particular, it has been shown that when the dimension is 1, the eigenfunctions are aways localized [MT61], see also [AW15] for a modern mathematical discussion.

where A^N is the deterministic adjacency matrix of a (finite) lattice in dimension d, and V^N is a diagonal matrix whose entries are random. The matrix A^N is the diffusion part, and V^N is the random potential part. Anderson fixed a disorder parameter $\lambda > 0$ and chose the diagonal entries of V^N to be uniform random variables in $[-\lambda, \lambda]$.

The localization phenomenon, and the presence or absence of transition, have been studied in many different models, including band matrices [FM91; Mir+96] or the sum of a Wigner and a diagonal matrix [LS13; LS16]. In Chapter 6, we shall be interested in adjacency matrices of random graphs. One of the simplest random graph model is the Erdős-Rényi model [ER59], in which we fix a number of vertices N and a parameter $p \in [0, 1]$, and

- we set an edge between two distinct vertices with probability p,
- the edges are all independent.

The localization phenomenon for the Erdős-Rényi model was studied in particular by Alt, Ducatez and Knowles in the series of articles [ADK23; ADK21b; ADK21a; ADK24]. They showed that there is localization-delocalization transition in the regime where

$$\sqrt{\ln N} \ll pN \leq \mathcal{O}(\ln N).$$

The eigenvectors associated to large eigenvalues where shown to be localized, and those associated to small eigenvalues where shown to be delocalized.

In a different regime, for an average degree close to constant order, i.e.

$$(\ln N)^{-1/9} \leq pN \leq (\ln N)^{1/40},$$

the eigenvectors corresponding to eigenvalues at the edge of the spectrum where shown to be localized by Hiesmayr and McKenzie [HM23], using a different approach from the one of Alt et al.

In Chapter 6, we consider only the localization result for a generalization of the Erdős-Rényi model, the Generalized Random Graph (GRG) model. In the GRG model, we fix a number of vertices N, and a sequence of weights $(w_x)_{x \in \{1,\dots,n\}}$. Then,

- we set an edge between two distinct vertices x and y with probability $\frac{w_x w_y}{\sum_{z=1}^N w_z + w_x w_y}$,
- the edges are all independent.

Hence, in the GRG model, the graph is inhomogeneous: vertices are biased to have a greater degree if they have a greater weight.

To state our result, we refine the notion of localization, and distinguish between localization and semi-localization. We consider the adjacency matrix of a GRG graph with N vertices and take the $N \to \infty$ limit. We will say that an eigenvector q is localized if its support is essentially of size bounded by a constant. More precisely, if there exists a normalized vector v with a support of constant size such that

$$|\langle q, v \rangle|^2 = 1 - o_{N\to\infty}(1).$$

We will say that q is semi-localized if there exists a set I of size $1 \ll \#I \ll N$ and a family of orthonormal vectors $(v_i)_{i\in I}$ such that

$$\sum_{i \in I} |\langle q, v_i \rangle|^2 = 1 - o_{N\to\infty}(1).$$

In our result, the actual order of $\#I$ will depend on the eigenvalue associated to the eigenvector q, and on the choice of weights.

An important feature in the case of the Erdős-Rényi model, in the regime studied by Alt et al., is that an eigenvector q associated to a large eigenvalue λ is localized around vertices x of degree D_x

close to λ^2. This fact is also true in the GRG model, in the regime we consider. With that in mind, we introduce the set

$$\mathcal{W}_{\lambda,\eta} = \left\{x \in \{1,\dots,N\} : \left|\sqrt{D_x} - \lambda\right| \leq \eta\right\} \qquad \text{for } \eta > 0.$$

The main result of Chapter 6 is then the following.

Theorem 1.7.1. *Let $\nu > 0$. There exists $C_\nu > 0$ such that for all N the following statement holds with probability $1 - \mathcal{O}(N^{-\nu})$. For every eigenvalue λ with associated normalized eigenvector q, and for every $\eta \leq |\lambda|/2$, we have*

$$\sum_{x \in \mathcal{W}_{\lambda,\eta}} \langle q, u_{\operatorname{sign}\lambda}(x)\rangle^2 \geq 1 - \frac{C_\nu}{\eta^2} \frac{\ln N}{\ln \ln N},$$

where $u_\pm(x)$ is a vector supported in $B_2(x)$, the ball centered on x of radius 2 for the graph distance, defined in Proposition 6.3.4.

Chapter 2

Preliminaries

We now define and discuss in some detail the notions briefly introduced in Chapter 1. We first give some important notation and definitions in Sections 2.1, 2.2, 2.3.2, and 2.4. We then discuss some techniques to study moments and cumulants of matrix models in Sections 2.5 and 2.6. An overview of the geometric notions used in Chapter 5 is proposed in Section 2.7. Section 2.8 is a discussion of the derivation of asymptotic expansion for matrix models. Finally, Section 2.9 discusses the techniques used in Chapter 6.

2.1 Partitions and permutations

We will describe the moments of a random matrix in terms of combinatorial objects, whose building blocks are partitions and permutations. We start by giving the notation pertaining to these objects.

We denote by $\mathbf{N} = \{0, 1, 2, \ldots\}$ the set of non-negative integers, and $\mathbf{N}^* = \{1, 2, \ldots\}$ the set of positive integers. For $n \in \mathbf{N}^*$, we denote by $[n]$ the set $\{1, 2, \ldots, n\}$. The cardinality of a set S is denoted by $\#S \in \mathbf{N} \cup \{+\infty\}$.

Definition 2.1.1. *Let S be a subset of $\mathbf{N}$. A* partition *of S is a set Π of nonempty subsets of S, called the* blocks *of Π, which are pairwise disjoint and such that*

$$\bigcup_{B \in \Pi} B = S.$$

We denote the set of partitions of S by $\mathcal{P}(S)$, and use the shorthand notation $\mathcal{P}_n := \mathcal{P}([n])$ for $n \in \mathbf{N}^*$.

Note that $\#\Pi$ is the number of blocks of the partition Π. Given an integer $n \in \mathbf{N}^*$, we define the two partitions of $[n]$:

$$0_n = \{\{i\} \colon i \in [n]\} \text{ and } 1_n = \{[n]\}.$$

The set of partitions $\mathcal{P}_n$ can be endowed with a partial order $\preceq$.

Definition 2.1.2. *Let $\Pi, \Pi' \in \mathcal{P}_n$. We say that Π is finer than Π', and write $\Pi \preceq \Pi'$ if for all $B \in \Pi$, there exists $B' \in \Pi'$ such that $B \subset B'$.*

We define the partition $\Pi \vee \Pi'$ as being the finest partition Π'' in $\mathcal{P}_n$ such that $\Pi \preceq \Pi''$ and $\Pi' \preceq \Pi''$.

We now turn to partitions of integers.

Definition 2.1.3. *Let $n \in \mathbf{N}^*$. A partition of the integer n is a non-increasing sequence of integers $\lambda = (\lambda_i)_{i \geq 1}$ such that*

$$n = \sum_{i=1}^{+\infty} \lambda_i.$$

We write $\lambda \vdash n$, and set $l(\lambda)$ to be the cardinality of the support of λ.

We denote by $\preceq$ the containment order, i.e. given λ, μ two partitions of integers, we write $\lambda \preceq \mu$ if

$$\lambda_i \leq \mu_i \text{ for all } i \in \mathbf{N}^*.$$

A partition with m_1 times the integer 1, m_2 times the integer 2, ..., m_l times the integer $l = \lambda_1$ is denoted by

$$(l^{m_l} \cdots 2^{m_2} 1^{m_1}).$$

Finally, we introduce notation pertaining to permutations. Let S be a subset of $\mathbf{N}$. We denote by $\mathfrak{S}(S)$ the set of permutations of the set S, that is bijections $S \to S$. Let $n \in \mathbf{N}^*$. We use the shorthand notation $\mathfrak{S}_n := \mathfrak{S}([n])$. The support of a permutation $\sigma \in \mathfrak{S}(S)$ is the set

$$\operatorname{Supp} \sigma = \{i \in S \colon \sigma(i) \neq i\} \subset S.$$

A permutation $\sigma \in \mathfrak{S}(S)$ acts naturally on S by

$$\begin{aligned} S &\to S \\ i &\mapsto \sigma \cdot i = \sigma(i). \end{aligned}$$

A subgroup G of $\mathfrak{S}(S)$ similarly acts on S. We say that G is *transitive* if G the natural action just defined is transitive: for each $i, j \in S$, there exists $\rho \in G$ such that $j = \rho(i)$. Given $\sigma_1, \ldots, \sigma_k \in \mathfrak{S}(S)^k$, the subgroup of $\mathfrak{S}(S)$ generated by $\sigma_1, \ldots, \sigma_k$ is denoted by

$$\langle \sigma_1, \ldots, \sigma_k \rangle .$$

A permutation $\sigma \in \mathfrak{S}_n$ may be written in product of disjoint cycles

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$$

for some $k \geq 1$. The permutations $\sigma_i, i \in [k]$ are permutations with disjoint support which are *cyclic*, i.e. $\langle \sigma_i \rangle$ seen as a subgroup of $\mathfrak{S}(\operatorname{Supp} \sigma_i)$ is transitive for all $i \in [k]$. Fix $i \in [k]$ and let $d_i = \# \operatorname{Supp} \sigma_i$. The permutation σ_i is written in cycle notation

$$\sigma_i = (j_1 \, j_2 \, \ldots \, j_{d_i}),$$

where j_1 is any element of $\operatorname{Supp} \sigma_i$ and $j_{m+1} = \sigma_i(j_m)$ with the convention that $j_{d_i+1} = j_1$. There are thus d_i equivalent ways to write this cycle. Most of the time, we shall write permutations using this notation. We write $\operatorname{Cycles}^*(\sigma)$ to denote the set of cycles of σ:

$$\operatorname{Cycles}^*(\sigma) = \{\sigma_1, \ldots, \sigma_k\} ,$$

and $\operatorname{Cycles}(\sigma)$ to denote the set of cycles of σ and fixed points:

$$\operatorname{Cycles}(\sigma) = \operatorname{Cycles}^*(\sigma) \cup (S \setminus \operatorname{Supp} \sigma) .$$

In that case, an element $i \in \operatorname{Cycles}(\sigma) \setminus \operatorname{Cycles}^*(\sigma)$ is seen as acting on S as the identity. With this notation, the elements of $\operatorname{Cycles}(\sigma)$ are in bijection with the orbits of the action of $\langle \sigma \rangle$ on S. These orbits define a partition of S, which we denote by 0_σ. The ordered family of length of the cycles of σ define a partition λ of the integer $\#S$. We say that λ is the cyclic type of σ. Given an integer $n \geq 1$ and a partition λ of n, we denote by $\mathcal{C}_\lambda$ the set of permutation in $\mathfrak{S}_n$ that are of cyclic type λ.

2.2 Observables, moments, and cumulants

2.2.1 Empirical measure and moments

We denote the set of $N \times N$ matrices with coefficients in a field $\mathbb{K}$ by $\mathrm{M}_N(\mathbb{K})$. Consider a family of random diagonalizable (say, Hermitian) matrices $(X^N)_{N \geq 1}$, with X^N of size $N \times N$ and sampled under a measure μ^N. In this Section, we denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ the N eigenvalues of X^N, taken with multiplicity. A way to study the spectrum of X^N in the large N limit is to consider the empirical measure of the eigenvalues of X^N, defined by

$$\hat{\nu}^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}, \tag{2.1}$$

and consider its weak convergence as $N \to \infty$. Note that it is a random measure as the eigenvalues $(\lambda_i)_{i\in[N]}$ are random. There are several ways to study $\hat{\nu}^N$. One way is to consider its Stieltjes transform:

$$\hat{W}(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\hat{\nu}^N(x)}{z-x} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\lambda_i} \text{ for } z \in \mathbb{C}\setminus\mathbb{R}.$$

It can be seen as the trace (up to a factor $1/N$) of the resolvent of X^N, $G(z) = (z - X^N)^{-1}$. The resolvent $G(z)$ will be used in Section 2.9 in the study of eigenvectors. Indeed, if $x_1, \ldots, x_N$ are the normalized eigenvectors of X^N, we have

$$G(z) = \sum_{i=1}^{N} \frac{1}{z-\lambda_i} x_i x_i^*.$$

An important technique in probability – see [Dia87] for instance – and more to the point, in random matrix theory, is the *moment method*. It was used by Wigner to prove its celebrated convergence theorem for Wigner matrices.

Theorem 2.2.1 (Wigner's theorem [Wig58]). *Let $(Y_{i,j})_{1\le i<j<\infty}$ and $(Z_i)_{i\in\mathbf{N}^*}$ be two infinite sequence of independent, identically distributed real random variables with unit variance, whose moments exists and are bounded by a constant independent of i, j. For all $N \ge 1$, let X^N be the $N \times N$ symmetric matrix defined by*

$$X^N_{i,j} = \begin{cases} Y_{i,j}/\sqrt{N} & \text{if } i \neq j, \\ Z_i/\sqrt{N} & \text{if } i = j. \end{cases}$$

Then, the empirical measure $\hat{\nu}^N$ of the eigenvalues of X^N converges weakly, in probability, to the standard semicircular distribution, given in term of the Lebesgue measure by

$$\mathrm{d}\sigma_{sc}(x) = \frac{\sqrt{4-x^2}}{2\pi}\mathbb{1}_{[-2,2]}(x)\mathrm{d}x.$$

For a modern discussion of this Theorem, see [AGZ10, Theorem 2.1.1]. This Theorem was proved by Wigner by computing the moments of the random matrix X^N, which we now define.

Definition 2.2.2 (Moments). *Let $N \in \mathbf{N}^*$. The moments of a single diagonalizable $N \times N$ random matrix X^N is the expectation of the moments of its empirical spectral distribution $\hat{\mu}^N$:*

$$m_k^N = \mathbb{E}\left[\hat{\nu}^N(x^k)\right].$$

Wigner compared the moments of a Wigner matrix X^N to those of the semicircular law, whose (usual) moments are the Catalan numbers:

$$m_{\text{sc},k} = \sigma_{\text{sc}}(x^k) = \begin{cases} \text{Cat}_{k/2} := \frac{1}{k+1}\binom{k}{k/2} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

The moments of the random matrix X^N can be re-expressed as:

$$m_k^N = \mathbb{E}\left[\hat{\nu}^N(x^k)\right] = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\lambda_i^k\right] = \frac{1}{N}\mathbb{E}\left[\text{Tr}\left[(X^N)^k\right]\right] = \mathbb{E}\left[\text{tr}\left[(X^N)^k\right]\right], \text{ for } k \ge 0.$$

In the expression above, Tr denotes the trace and tr is the normalized trace:

$$\mathrm{tr} = \frac{1}{N} \mathrm{Tr}\,. \tag{2.2}$$

We are going to follow the moment approach in most of the sequel. Its interest in random matrix theory mostly lies on the fact that random matrix models are often specified in terms of their entries. While studying the expression of the eigenvalues in terms of the entries is usually intractable, the moments can easily be expressed in terms of the entries: the empirical moments coincide with the traces of powers of X^N.

We also consider the *joint moments* of X^N. Let λ be a partition of an integer. We define for a matrix M,

$$\mathrm{Tr}_\lambda(M) = \prod_{i=1}^{l(\lambda)} \mathrm{Tr}\Big(M^{\lambda_i}\Big) \quad \text{and} \quad \mathrm{tr}_\lambda(M) = \prod_{i=1}^{l(\lambda)} \mathrm{tr}\Big(M^{\lambda_i}\Big).$$

The joint moments are then the quantities

$$\mathbb{E}\left[\mathrm{Tr}_\lambda(X^N)\right],$$

for all partitions of integers λ.

This approach generalizes in the case of several random matrices. Let $m \geq 1$, and assume that $X_1^N, \dots, X_m^N$ are $N \times N$ Hermitian (not necessarily independent) random matrices. The joint moments of these matrices are defined using the notion of non-commutative polynomial.

Definition 2.2.3 (Non-commutative polynomial)**.** *The algebra of non-commutative polynomials in m indeterminates is the free $\mathbb{K}$-algebra generated by m elements $x_1, \dots, x_m$. We denote this algebra by*

$$\mathbb{K}\langle x_1, \dots, x_m\rangle\,.$$

A monic monomial in $\mathbb{K}\langle x_1, \dots, x_m\rangle$ is an element of the free monoid with m generators $x_1, \dots, x_m$.

Said otherwise, a monic monomial is a word in the letters $x_1, \dots, x_m$ or the identity, and a non-commutative polynomial is a linear combination of such monomials.

Given $P \in \mathbb{C}\langle x_1, \dots, x_m\rangle$, and m matrices $X_1^N, \dots, X_m^N$ of size $N \times N$, we can evaluate P in $X_1, \dots, X_m$ by replacing each letter x_i in P by X_i^N and the empty word by Id_N, the identity matrix of size N. Formally, we consider the representation of $\mathbb{K}\langle x_1, \dots, x_m\rangle$ (that is the algebra morphism $\mathbb{K}\langle x_1, \dots, x_m\rangle \to \mathrm{M}_N(\mathbb{K})$) determined by

$$\rho\colon x_i \mapsto X_i^N, \text{ for all } i \in [m].$$

and set

$$P(X_1^N, \dots, X_m^N) = \rho(P) \in \mathrm{M}_N(\mathbb{K}).$$

Analogous notions to those available for commutative polynomials can be defined. In particular, we have a notion of degree.

Definition 2.2.4 (Degree)**.** *Let $P \in \mathbb{C}\langle x_1, \dots, x_m\rangle$ be a monomial. The degree of P with respect to x_i is the number of occurrences of the letter x_i in P. We denote it by $\deg_{x_i} P$.*

The moments and joint moments are then defined using the following notation. For any tuple of matrices $M_1, \dots, M_n \in \mathrm{M}_N(\mathbb{K})$ and permutation $\sigma \in \mathfrak{S}_n$, we define the generalized trace

$$\mathrm{Tr}_\sigma(M_1, \dots, M_n) = \sum_{i\colon [n]\to[N]} \prod_{p=1}^{n} (M_p)_{i(p),i(\sigma(p))} = \prod_{\substack{\pi\in\mathrm{Cycles}(\sigma)\\ \pi=(u_1\,\dots\,u_{|\pi|-1})}} \mathrm{Tr}\left(M_{u_1}M_{u_2}\cdots M_{u_{|\pi|-1}}\right).$$

We also define the associated normalized trace

$$\operatorname{tr}_\sigma(M_1,\ldots,M_n) = \frac{1}{N^{\#\sigma}} \operatorname{Tr}_\sigma(M_1,\ldots,M_n).$$

The joint moments of the random matrices $X_1^N,\ldots,X_m^N$ are then the numbers

$$\mathbb{E}\left[\operatorname{tr}_\sigma(X_{i(1)}^N,\ldots X_{i(n)}^N)\right],$$

with $n \geq 1$, $i\colon [n] \to [m]$, and $\sigma \in \mathfrak{S}_n$.

2.2.2 Cumulants, connected and disconnected objects

Rather than studying the moments of our random matrices, it will prove easier to study their cumulants instead. Cumulants were introduced by the Danish astronomer Thiele at the end of the XIXth century (see for instance [Hal00]). They contain the same information as moments but are often easier to work with – especially when the moments have a combinatorial interpretation.

Definition 2.2.5 (Cumulants). *Let $Z_1,\ldots,Z_m$ be complex random variables whose joint moments*

$$\mathbb{E}\left[Z_1^{k_1}\cdots Z_m^{k_m}\right], k_1,\ldots,k_m \geq 0$$

are all finite. Let $l \geq 1$ and $i\colon [l] \to [m]$. The joint cumulants are defined inductively by

$$\kappa_l(Z_{i(1)},\ldots,Z_{i(l)}) = \mathbb{E}\left[Z_{i(1)}\cdots Z_{i(l)}\right] - \sum_{\substack{\Pi\in\mathcal{P}_l\\ \#\Pi\geq 2}} \prod_{B\in\Pi} \kappa_{\#B}(Z_{i(p)}; p\in B). \tag{2.3}$$

Notice that κ is a symmetric multilinear form, hence the term $\kappa_{\#B}(Z_{i(p)}; p \in B)$ in (2.3) is well-defined. As it is a multilinear form, it will be convenient to use the tensor notation in the sequel: for $l_1,\ldots,l_m \geq 0$ and $l = \sum_i l_i$, we write

$$\kappa_l\left(Z_1^{\otimes l_1},\ldots,Z_m^{\otimes l_m}\right)$$

to mean

$$\kappa_l\left(\underbrace{Z_1,\ldots,Z_1}_{l_1\text{ times}},\ldots,\underbrace{Z_m,\ldots,Z_m}_{l_m\text{ times}}\right).$$

Example 2.2.6. The first three cumulants are

$$\begin{aligned}
\kappa_1(Z) &= \mathbb{E}[Z],\\
\kappa_2(Z_1,Z_2) &= \mathbb{E}[Z_1Z_2] - \mathbb{E}[Z_1]\,\mathbb{E}[Z_2],\\
\kappa_3(Z_1,Z_2,Z_3) &= \mathbb{E}[Z_1Z_2Z_3] - \sum_{i=1}^{3}\mathbb{E}\left[\prod_{\substack{j=1\\ j\neq i}}^{3} Z_j\right]\mathbb{E}[Z_i] + 2\mathbb{E}[Z_1]\,\mathbb{E}[Z_2]\,\mathbb{E}[Z_3].
\end{aligned}$$

There are several alternative definitions of the cumulants. One is

$$\kappa_l(Z_{i(1)},\ldots,Z_{i(l)}) = \sum_{\Pi\in\mathcal{P}_l} (-1)^{\#\Pi-1}(\#\Pi-1)! \prod_{B\in\Pi} \mathbb{E}\left[\prod_{p\in B} Z_{i(p)}\right], \tag{2.4}$$

another is

$$\kappa_l(Z_{i(1)},\ldots,Z_{i(l)}) = \frac{\partial^l}{\partial t_1\cdots\partial t_l} \ln \mathbb{E}\left[\exp\left(\sum_{p=1}^{l} t_p Z_{i(p)}\right)\right]\Big|_{t_1=\cdots=t_l=0}. \tag{2.5}$$

See for instance the review by Speed [Spe83] for a proof of the equivalence of these definitions. Cumulants are best understood in the context of Möbius inversion theory pioneered by Rota in his seminal article [Rot64]. Möbius inversion appears as a general way to understand the principle of *inclusion-exclusion* that appears frequently in combinatorics. In particular, the moment-cumulant relations (2.3) and (2.4) appear in the theory of Möbius inversion in the case of the lattice of partitions of $[l]$.

Those inversion formulae can be though of as relating numbers counting connected and disconnected objects. Fix an integer $l \geq 1$ and consider a family of numbers indexed by subsets of $[l]$, $(c_S)_{S\subset[l]}$, thought of as numbers of connected objects. An analogue of formula (2.3) allows us to define number of disconnected objects: for all $B \subset [l]$

$$d_B = \sum_{\Pi\in\mathcal{P}(B)} \prod_{S\in\Pi} c_S, \tag{2.6}$$

where the sum on Π describes all the way to cut a disconnected object into connected objects. This relation can be inverted using an analogue of (2.4):

$$c_S = \sum_{\Pi\in\mathcal{P}(S)} (-1)^{\#\Pi-1}(\#\Pi-1)! \prod_{B\in\Pi} d_B. \tag{2.7}$$

We can rewrite those relations in terms of (formal) generating series in an infinite number of parameters $\boldsymbol{t} = (t_i)_{i\in\mathbf{N}^*}$. We set

$$C(\boldsymbol{t}) = \sum_{l\geq 0} \frac{1}{l!} \sum_{\substack{S\subset\mathbf{N}^* \\ \#S=l}} \left(\prod_{i\in S} t_i\right) c_S, \text{ and } D(\boldsymbol{t}) = \sum_{l\geq 0} \frac{1}{l!} \sum_{\substack{S\subset\mathbf{N}^* \\ \#S=l}} \left(\prod_{i\in S} t_i\right) d_S,$$

and (2.7) becomes

$$C(\boldsymbol{t}) = \ln D(\boldsymbol{t}).$$

which corresponds to (2.5). See the book of Flajolet and Sedgewick [FS13] for a thorough discussion of this kind of transformations.

2.2.3 Free probability theory

In the multi-matrix case, a powerful framework for the study of the moments is free probability theory. It was introduced by Voiculescu to study operator algebras and soon came to be of interest in random matrix theory [Voi91]. It is the study of non-commutative probability spaces, together with the notion of *freeness*, loosely speaking a non-commutative analogue of independence. We only give a quick introduction to free probability theory. For a more complete discussion, see for instance [MS17].

Definition 2.2.7 (Non-commutative probability space)**.** *Let $\mathcal{A}$ be a unital $\mathbb{C}$-algebra and τ a linear map $\mathcal{A} \to \mathbb{C}$ satisfying $\tau(1) = 1$. We say that the pair $(\mathcal{A}, \tau)$ is a non-commutative probability space. The elements of the algebra $\mathcal{A}$ are* non-commutative random variables.

If $\mathcal{A}$ is a $*$-algebra and τ satisfies $\tau(a^*a) \geq 0$ for all $a \in \mathcal{A}$, we say that τ is a state and that $(\mathcal{A}, \tau)$ is a $*$-probability space.

We shall consider two natural $*$-probability spaces.

Example 2.2.8. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $N \in \mathbf{N}^*$. Denote by $L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{C})$ the set of complex random variables having all their moments finite. Define

$$\mathcal{A} = \mathrm{M}_N(\mathbb{C}) \otimes_{\mathbb{C}} L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{C}).$$

It is a unital $*$-algebra (the involution being the complex transposition). We set

$$\tau = \mathbb{E}\left[\mathrm{tr}\left(\cdot\right)\right].$$

The Cauchy-Schwarz inequality implies that τ is a *faithful* state, i.e. $\tau(a^*a) = 0$ if and only if $a = 0$. Hence, $(\mathcal{A}, \tau)$ is a $*$-probability space.

Example 2.2.9. Let $m \in \mathbf{N}^*$. Consider the algebra $\mathcal{A} = \mathbb{C}\langle x_1, \ldots, x_m, x_1^*, \ldots, x_m^* \rangle$, endowed with the involution $*$ defined by $(x_i)^* = x_i^*$ and for any $P, Q \in \mathcal{A}$, $(PQ)^* = Q^* P^*$. This makes $\mathcal{A}$ into a $*$-algebra. For any state τ, $(\mathcal{A}, \tau)$ is a $*$-probability space.

Definition 2.2.10 (Convergence in distribution). *Let $(\mathcal{A}_N, \tau_N)_{N \geq 1}$ be a sequence of non-commutative probability spaces, and $(\mathcal{A}, \tau)$ be a non-commutative probability space. Fix an index set I and for all $N \in \mathbf{N}^*$, a family $(a_i^N)_{i \in I} \in \mathcal{A}_N^I$. We say that $(a_i^N)_{i \in I}$ converges in probability to $(a_i)_{i \in I} \in \mathcal{A}^I$ if for all $k \geq 1$ and $j \colon [k] \to I$ we have*

$$\lim_{N \to \infty} \tau_N \left(a_{j(1)}^N \cdots a_{j(k)}^N \right) = \tau \left(a_{j(1)} \cdots a_{j(k)} \right).$$

We denote this by $(a_i^N)_{i \in I} \xrightarrow[N \to \infty]{} (a_i)_{i \in I}$.

When we consider the convergence of the joint moments in multi-matrix models, we are considering the convergence in distribution of a sequence of non-commutative random variables in the spaces of Example 2.2.8 to variables in a space of the form of Example 2.2.9.

Definition 2.2.11 (Freeness). *Let $n \in \mathbf{N}^*$. Let $(\mathcal{A}, \tau)$ be a non-commutative probability space, and $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be unital sub-algebras of $\mathcal{A}$. We say that $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are free with respect to τ if for all integer $m \geq 2$, function $j \colon [m] \to [n]$, and elements $a_1, \ldots, a_m \in \mathcal{A}$ satisfying*

- $\tau(a_i) = 0$ *for all* $i \in [m]$,
- $a_i \in \mathcal{A}_{j(i)}$ *for all* $i \in [m]$,
- $j(1) \neq j(2), j(2) \neq j(3), \ldots, j(m-1) \neq j(m)$,

we have

$$\tau(a_1 a_2 \cdots a_m) = 0. \tag{2.8}$$

If we unpack the identity (2.8), it means that in presence of freeness, we can express the value of τ on any product in terms of product of values of τ evaluated in produces involving elements of only one sub-algebra.

In this context, the notion of freeness can be understood as a way to compute the joint law of several random variables from their marginals: by computing the moments involving any number of non-commutative random variables from moments involving only one non-commutative random variable. We do not have freeness in general, but Voiculescu's theorem allow us to obtain freeness in the large N limit. The relevant statement for random matrix theory can be found in [MS17, Chapter 4, Theorem 9].

Theorem 2.2.12 (Voiculescu [Voi90]). *Let $(A^N)_{N \geq 1}$ and $(B^N)_{N \geq 1}$ be two sequences of deterministic complex matrices, and (U^N) a sequence of Haar-distributed unitary matrices, with A^N, B^N, and U^N of size $N \times N$.*

If $A^N \xrightarrow[N \to \infty]{} a$ and $B^N \xrightarrow[N \to \infty]{} b$, then

$$(A^N, U^N B^N (U^N)^*) \xrightarrow[N \to \infty]{} (a, b),$$

where a, b are free. We say that A^N and $U^N B^N (U^N)^*$ are asymptotically free.

This is the version of Voiculescu's theorem that we shall be interested in in Section 3.4. There exists several similar asymptotic freeness results, for instance between sequences of deterministic and Wigner matrices.

Freeness is governed by the combinatorics of non-crossing partitions [MS17].

Definition 2.2.13 (Non-crossing partition). *Let $n \in \mathbf{N}^*$. A non-crossing partition of $[n]$ is a partition Π of $[n]$ such that there are no quadruplet $1 \leq i < j < k < l \leq n$ such that $\{i,k\}$ and $\{j,l\}$ are subset of distinct blocks.*

We shall not use non-crossing partitions in the sequel but rather maps, informally graphs drawn on surfaces. However, we remark that the two points of view are in many cases equivalent. In particular, non-crossing partitions correspond to maps on the sphere with one vertex.

In fact, the notion of freeness can be enriched by the notion of higher order freeness, developed by Mingo and Speicher [MS06; MŚS07], and Collins, Mingo, Śniady, and Speicher [Col+07]. Higher order freeness comes with its own combinatorial objects, partitioned permutations. A description of the functional relations appearing in higher order freeness, and a description of sub-leading order terms, was given by Borot, Charbonnier, Garcia-Failde, Leid, and Shadrin [Bor+21]. A salient point of their work is the key role played by monotone Hurwitz numbers, which will appear in Chapter 3.

2.3 Symmetries and invariant models

2.3.1 Groups of symmetries

Dyson argued that three types of groups are singled out from physical arguments [Dys62b]: the orthogonal, unitary and symplectic groups. This "Threefold way" follows from Frobenius Theorem.

Theorem 2.3.1 (Frobenius Theorem). *There are three associative division algebras over the real numbers:*

- *the real numbers $\mathbb{K}_1 = \mathbb{R}$,*
- *the complex numbers $\mathbb{K}_2 = \mathbb{C}$, and*
- *the quaternions $\mathbb{K}_4$.*

The Threefold way is at the root of the definition of most of the matrix models we will consider in this Thesis. Fix a matrix size N. The three groups we discuss are the groups $\mathbb{U}_\beta$ for $\beta \in \{1,2,4\}$ defined by

$$\mathbb{U}_\beta(N) = \{U \in \mathrm{M}_N(\mathbb{K}_\beta) \colon UU^* = U^*U = \mathrm{Id}_N\}.$$

Here, the involution $*$ is the operation of conjugation and transposition: U^* is the transpose of U in the case $\beta = 1$, the usual conjugate transpose in the case $\beta = 2$, and the quaternionic-conjugate transpose in the case $\beta = 4$. It will be convenient in the sequel to write $\mathbb{U}(N) = \mathbb{U}_2(N)$ to denote the usual unitary group.

We recall that these three groups are compact Lie groups, and thus each of them admits a Haar measure. The Haar measure on a compact Hausdorff continuous group G is the unique (up to rescaling) measure on the Borel subset of G the that is invariant under multiplication on the left or on the right by an element of G, see for instance [Fol95, Section 2.2].

The three groups $\mathbb{U}_\beta$ appear as symmetry groups of the vector spaces of symmetric $\mathcal{H}_1(N)$, Hermitian $\mathcal{H}_2(N)$, and quaternionic self-adjoint $\mathcal{H}_4(N)$ matrices of size $N \times N$. We define them by

$$\mathcal{H}_\beta(N) = \{M \in \mathrm{M}_N(\mathbb{K}_\beta) \colon M = M^*\} \text{ for } \beta \in \{1,2,4\}.$$

Fix $\beta \in \{1,2,4\}$, $U \in \mathbb{U}_\beta(N)$, and $H \in \mathcal{H}_\beta(N)$. The group $\mathbb{U}_\beta(N)$ acts on $\mathcal{H}_\beta(N)$ by conjugation:

$$U \cdot H = UHU^* \in \mathcal{H}_\beta(N).$$

Remark 2.3.2 (Conjugation-invariant observables). In the presence of conjugation invariance, the joint moments are the natural polynomial observables. Indeed, let $n \geq 1$ be an integer and $O \colon X \in \mathrm{M}_N(\mathbb{C}) \mapsto O(X) \in \mathbb{C}$ be a monic (commutative) polynomial in the entries of X of degree n that is unitarily invariant, i.e. for all $X \in \mathrm{M}_N(\mathbb{C})$ and all $U \in \mathbb{U}_2(N)$, we have

$$O(UXU^*) = O(X).$$

In that case, a result of geometric invariant theory, see [Pro76], implies that there exists a partition λ of n, such that

$$O(X) = \mathrm{Tr}_\lambda(X).$$

Similarly, let O be a monic polynomial of degree n in the entries of m matrices $X_1, \dots, X_m$ that is unitarily invariant, i.e. for all $X_1, \dots, X_m \in \mathrm{M}_N(\mathbb{C})$ and $U \in \mathbb{U}_2(N)$,

$$O(UX_1U^*, UX_2U^*, \dots, UX_nU^*) = O(X_1, X_2, \dots, X_n).$$

Then, there exists a mapping $i\colon [n] \to [m]$ and a permutation $\sigma \in \mathfrak{S}_n$ such that

$$O(X_1, \dots, X_m) = \mathrm{Tr}_\sigma(X_{i(1)}, \dots, X_{i(n)}).$$

2.3.2 $\mathbb{U}_\beta(N)$-invariant matrix models

One-matrix models

Fix an integer $N \geq 1$. The first invariant random matrix models were models involving a single random matrix $X \in \mathcal{H}_\beta(N)$, whose law is invariant by conjugation by one of the group $\mathbb{U}_\beta(N)$ defined in Section 2.3.1: for a given $\beta \in \{1, 2, 4\}$ it is assumed that for all $U \in \mathbb{U}_\beta(N)$ the matrices X and UXU^* have the same law. If we further assume that the entries of X are – up to the self-adjointness – independent, there is only one possible law, up to rescaling of the matrix. This is explained in details in the book of Mehta [Meh04], for instance. This law is called the Gaussian orthogonal, unitary, or symplectic ensemble depending on whether $\beta = 1$, 2, or 4 (abbreviated as GOE, GUE, or GSE). It corresponds to X having entries which are distributed according to the real, complex or quaternionic normal law. We write this measure as

$$\mathrm{d}\mu^N_{\beta,0}(X) = \frac{1}{Z^N_{\beta,0}} \exp\Big(-\frac{\beta N}{4} \mathrm{Tr}\big(X^2\big)\Big) \mathrm{d}X, \tag{2.9}$$

where $\mathrm{d}X$ denotes the Lebesgue measure on $\mathcal{H}_\beta(N)$, and Z^N_0 is the partition function

$$Z^N_{\beta,0} = \int_{\mathcal{H}_\beta(N)} \exp\Big(-\frac{\beta N}{4} \mathrm{Tr}\big(X^2\big)\Big) \mathrm{d}X = \left(\frac{4\pi}{\beta N}\right)^{\beta N^2/4 + (1-\beta/2)N/2}.$$

In the sequel, we consider the family of probability measures on $\mathcal{H}_\beta(N)$ of the form

$$\mu^N_{\beta,V}(X) = \frac{1}{Z^N_{\beta,V}} \exp\Big(-\frac{\beta N}{2} \mathrm{Tr}\left[V(X) + \frac{X^2}{2}\right]\Big) \mathrm{d}X, \tag{2.10}$$

with $Z^N_{\beta,V}$ the corresponding partition function, and $V \in \mathbb{C}\langle x \rangle$ a polynomial which is *confining*.

Definition 2.3.3 (Confining potential)**.** *Let $\beta > 0$. Let $\mathcal{V}\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. We say that $\mathcal{V}$ is confining if*

$$\liminf_{|x|\to\infty} \frac{\mathcal{V}(x) + x^2/2}{\beta \ln |x|} > 1.$$

In the particular case where V is a polynomial, it being confining means that $V(X) + X^2/2 \to \infty$ as $\|X\| \to \infty$. Note that $\mathrm{Tr}\, V(X) = \mathrm{Tr}\, V(UXU^*)$ for any $U \in \mathbb{U}_\beta(N)$: μ^N_V is thus invariant by the action of $\mathbb{U}_\beta(N)$.

A random matrix X distributed according to $\mu^N_{\beta,V}$ can be diagonalized as

$$X = UDU^*,$$

with U a random element of $\mathbb{U}_\beta(N)$ and $D = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$ a random diagonal matrix whose coefficients $\lambda_1, \dots, \lambda_N$ are the N eigenvalues of X (taken with multiplicities). The invariance of $\mu^N_{\beta,V}$ under action of $\mathbb{U}_\beta(N)$ implies that U is Haar-distributed, see for instance [AGZ10, Corollary 2.5.4]. The joint law of the eigenvalues of X is as follows.

Theorem 2.3.4 (See for instance [AGZ10, Theorem 2.5.2])**.** *Let $N \geq 1$, $\beta \in \{1,2,4\}$, and $V \in \mathbb{C}\langle x \rangle$ be a confining potential. The joint law of the eigenvalues of X, distributed according to $\mu^N_{\beta,V}$, is a measure $\nu^N_{\beta,V}$ absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^N$, whose density is*

$$\frac{\mathrm{d}\nu^N_{\beta,V}}{\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N}(\lambda_1,\ldots,\lambda_N) = \frac{1}{\mathcal{Z}^N_{\beta,V}} |\Delta(\lambda_1,\ldots,\lambda_N)|^\beta \exp\left(-\frac{\beta N}{2}\sum_{i=1}^N \left(V(\lambda_i) + \frac{\lambda_i^2}{2}\right)\right), \tag{2.11}$$

with $\mathcal{Z}^N_{\beta,V}$ the appropriate normalization constant.

The β-ensemble

The joint law (2.11) of the eigenvalues of a $\mathbb{U}_\beta(N)$-invariant random matrix can straightforwardly be generalized to all $\beta > 0$. This gives a measure on $\mathbb{R}^N$ called the β-ensemble.

Definition 2.3.5 (β-ensemble)**.** *Let $N \geq 1$ be an integer and $\beta > 0$. The β-ensemble is the measure $\nu^N_{\beta,V}$ on $\mathbb{R}^N$ that is absolutely continuous with respect to the Lebesgue measure, and whose density is given by* (2.11).

A natural question follows:

Question 2.3.6. When $\beta \notin \{1,2,4\}$, is $\nu^N_{\beta,0}$ the joint law of eigenvalues of a random matrix with independent entries?

This question was answered positively by Dumitriu and Edelman [DE02]. We postpone the description of this matrix model until Chapter 4, where it will be used. Note however, that for $\beta \notin \{1,2,4\}$, there is no natural group of symmetries similar to $\mathbb{U}_1(N)$, $\mathbb{U}_2(N)$, or $\mathbb{U}_4(N)$.

Multi-matrix models

In Chapter 3, we will consider the more general case of models involving several random matrices $X_1,\ldots,X_m$ for $m \geq 1$, possibly not independent. We will consider models in which these random matrices are jointly $\mathbb{U}_\beta$-invariant, that is for all $U \in \mathbb{U}_\beta$:

$$(X_1,\ldots,X_m) \text{ and } (UX_1U^*,\ldots,UX_mU^*) \text{ are identically distributed.} \tag{2.12}$$

The following discussion applies in the three cases $\beta \in \{1,2,4\}$. However, the case of interest in Chapter 3 is $\beta = 2$, and we concentrate on this. Let us define the family of models we are interested in. Let $V \in \mathbb{C}\langle X_1,\ldots,X_m\rangle$ be such that $\mathrm{Tr}\, V(X_1,\ldots,X_m) \to \infty$ as $\max_{i\in[m]} \|X_i\| \to \infty$. We define the measure μ^N_V on $\mathcal{H}_2(N)^m$ by

$$\mu^N_V(X_1,\ldots,X_m) = \frac{1}{Z^N_V}\exp\left(-N\,\mathrm{Tr}\left[V(X_1,\ldots,X_m) + \sum_{i=1}^m \frac{X_i^2}{2}\right]\right)\mathrm{d}X_1\cdots\mathrm{d}X_m. \tag{2.13}$$

The particular case $V = 0$ corresponds to the law of m independent GUE matrices. A way to study this kind of models is to proceed as in Section 2.3.2 and diagonalize the matrices. We write the potential V as a sum of one-matrix potentials $\hat{V}_i \in \mathbb{C}\langle x_i\rangle$, $i \in [m]$ and an interaction potential $W \in \mathbb{C}\langle x_1,\ldots,x_m\rangle$:

$$V(x_1,\ldots,x_m) = \sum_{i=1}^m \hat{V}_i(x_i) + W(x_1,\ldots,x_m),$$

and set $\hat{V}(x_1,\ldots,x_m) = \sum_i \hat{V}_i(x_i)$. We assume that on $\mathcal{H}_2(N)$, $\mathrm{Tr}\, V$ takes real values only and that for each $i \in [m]$, $\hat{V}_i$ is confining with degree satisfying $\deg_{x_i} W < \deg_{x_i} \hat{V}_i$. This ensures that the measure is well-defined. For any μ_V^N-integrable function $f\colon \mathcal{H}_2(N)^m \to \mathbb{C}$ of the m random matrices we have after diagonalizing

$$\mu_V^N(f) = \frac{\mu_{\hat{V}}^N\left[\left(f\mathrm{e}^{-N\operatorname{Tr}W}\right)(X_1,\ldots,X_m)\right]}{\mu_{\hat{V}}^N\left[\mathrm{e}^{-N\operatorname{Tr}W(X_1,\ldots,X_m)}\right]} = \frac{\mathbb{E}\left[\left(f\mathrm{e}^{-N\operatorname{Tr}W}\right)(U_1D_1U_1^*,\ldots,U_mD_mU_m^*)\right]}{\mathbb{E}\left[\mathrm{e}^{-N\operatorname{Tr}V(X_1,\ldots,X_m)}\right]},$$

where in the last expectation $\mathbb{E}$: for all $i \in [m]$, $D_i = \operatorname{diag}(\lambda_{i,1},\ldots,\lambda_{i,N})$ with $(\lambda_{i,j})_{j\in[N]}$ distributed according to $\nu_{\hat{V}_i}^N$, and U_i a Haar-distributed $N\times N$ unitary matrix. The matrices $D_1,U_1,\ldots,D_m,U_m$ are independent.

We may perform the integration on the unitary matrices before the integration on the diagonal matrices. This leads naturally to the problem of estimating unitary integrals, which we study in Chapter 3. Let us detail this. We can assume that the diagonal matrices $(D_i)_{i\in[m]}$ are deterministic, and consider the new probability measure proportional to

$$\exp(-NW(U_1D_1U_1^*,\ldots,U_mD_mU_m^*))\mathrm{dHaar}^{\otimes m}(U_1,\ldots,U_m).$$

As we shall see in Chapter 3, under some hypotheses, we can express cumulants under this measure in terms of the joint moments of the matrices $D_1,\ldots,D_m$ only, i.e. quantities $\operatorname{tr}\big(D_i^k\big)$ for $i\in[m]$ and $k\geq 0$. Said otherwise, we can compute all the moments involving all m matrices $X_1,\ldots,X_m$ in terms of the moments of the marginals. The latter moments, each involving only one random matrix, are easier to study than moments of multi-matrix models.

2.4 Maps and coverings

2.4.1 Maps

Informally, the discrete surfaces we consider are graphs drawn on surfaces. They were much studied by Tutte, who developed an enumerative theory of planar maps [Tut62a; Tut62b; Tut62c; Tut63]. Among many other results, he obtained a closed expression for the number of maps on the sphere with a given number of edges [Tut68].

By surface, we mean a compact topological manifold of dimension 2. A simple curve in a surface S is a continuous mapping $\gamma\colon [0,1]\to S$ such that $\gamma|_{(0,1)}$ is injective.

Definition 2.4.1 (Embedded graph)**.** *Let $\Gamma=(V,E)$ be a graph with vertex set V and edge set E. This graph may have loops and multiple edges. Let S be a connected compact surface. A graph embedding of Γ in S is a mapping ι such that:*

- *for all vertex $v\in V$, $\iota(v)$ is a point on S, and all such points are distinct;*
- *for all edge $e\in E$ with endpoints v_1 and v_2, $\iota(e)$ is a continuous simple curve γ_e from $\gamma_e(0)=\iota(v_1)$ to $\gamma_e(1)=\iota(v_2)$, and all such curves may only intersect at their endpoints;*
- *each connected component of $S\setminus\big(\iota(V)\cup\bigcup_{e\in E}\iota(e)([0,1])\big)$ is homeomorphic to a disc.*

The triple (S,Γ,ι) is said to be an embedded graph.

Definition 2.4.2 (Embedded graph isomorphism)**.** *Let (S,Γ,ι) and (S',Γ',ι') be two embedded graphs. We say they are isomorphic if there exists an orientation-preserving homeomorphism $\phi\colon S\to S'$ and a graph isomorphism $\psi\colon\Gamma\to\Gamma'$ such that*

$$\iota = \phi^{-1}\circ\iota'\circ\psi.$$

Definition 2.4.3 (Map)**.** *A map is an isomorphism class of embedded graphs (in the sense of Definition 2.4.2). The genus of a map is the genus of the underlying surface of any of its representative.*

The genus of a surface is a topological invariant so the genus of a map is well-defined. An important topological formula that we will use is Euler's formula.

Theorem 2.4.4 (Euler's formula, see for instance [Spa81]). *Let* $\mathfrak{m}$ *be a map with* V *vertices,* E *edges, and* F *faces. We have*

$$2 - 2g = V - E + F.$$

Remark 2.4.5. We said earlier that a surface was a continuous topological manifold. In fact, we could have restricted the discussion to smooth manifold, or in the case of an orientable map, to Riemann surfaces. The main point is that compact topological manifold of dimension 2 are well-behaved: they are triangulable – see [MT01, Theorem 3.1.1] – and can be equipped with a smooth atlas or, it the orientable case, with a holomorphic atlas – see for instance [Rey89, Théorème 2.2].

Remark 2.4.6. In the sequel, we shall introduce maps with additional labellings (of the edges or vertices for instance). In that case, we change the notion of isomorphism as follow: two embedded graphs with labelling are isomorphic if there exists an orientation-preserving homeomorphism $\phi\colon S \to S'$ and a graph isomorphism $\psi\colon \Gamma \to \Gamma'$ *which preserves the labelling* such that

$$\iota = \phi^{-1} \circ \iota' \circ \psi.$$

We immediately make use of Remark 2.4.6 and introduce maps with colored vertices.

Definition 2.4.7 (Hypermap). *A hypermap is a map whose vertices are colored either white or black and such that each edge is incident to exactly one white vertex and one black vertex. A black vertex may be referred to as a hyperedge.*

Equivalently,a hypermap is a bipartite map with a bi-coloration of the vertices. Every hypermap can be seen as a map by erasing the coloring of the vertices. Conversely, we may see a map as a hypermap by coloring all the vertices white, and adding a black vertex in the middle of each edge. The edges of the hypermap thus created will be referred to as the *half-edges* of the map.

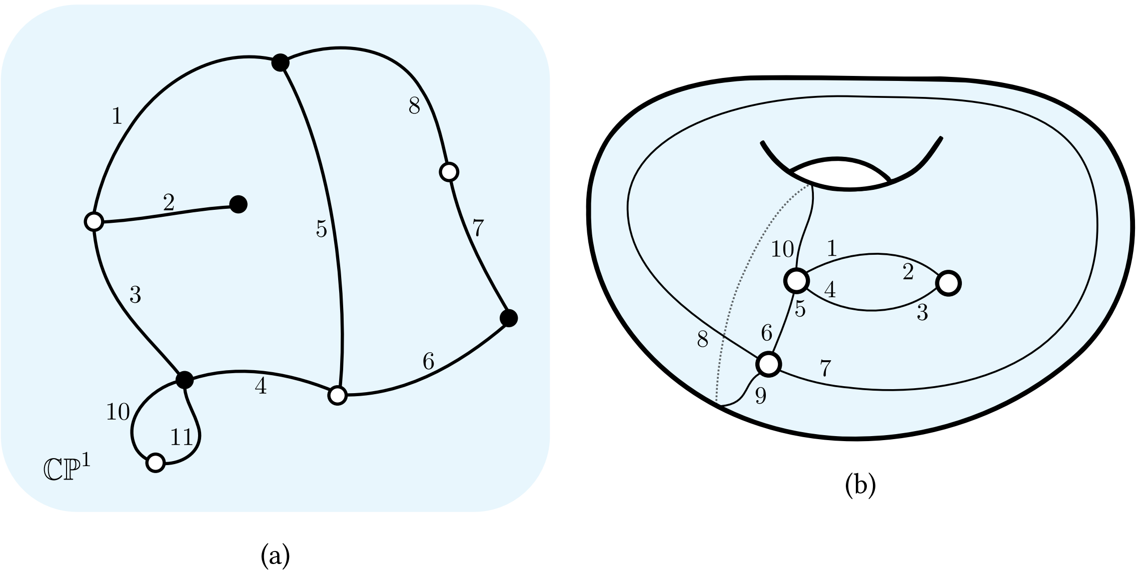


Figure 2.1: (a) a planar hypermap with labelled edges, (b) a map on the torus with labelled half-edges.

Definition 2.4.8. *The degree of a face or vertex is the number of half-edges incident to it. The face or vertex profile of a map with* n *half-edges is a partition* λ *of* n *such that each part of* λ *is the degree of a distinct face or vertex of the map.*

In the case of orientable maps, we orient each edge of a hypermap from its white vertex to its black vertex. We thus distinguish a left side and a right side of each edge. We say that an edge e (or a half-edge, in the case of a map) is incident to a face f if f is at the left of e. Note that it is no longer possible to canonically distinguish the left and right side of an edge in a non-orientable map.

We now make use of Remark 2.4.6 to consider orientable bipartite maps with edges labelled by integers. We now give a way to describe those maps using permutations.

Definition 2.4.9. *Let $n \geq 1$ be an even integer. A combinatorial hypermap is a pair of permutations $(\sigma, \alpha) \in \mathfrak{S}_n$ such that $\langle \sigma, \alpha \rangle$ is transitive. We say it is a combinatorial map if α is an involution without fixed point, i.e.*

$$\alpha \in \mathcal{I}_n \coloneqq \left\{ \rho \in \mathfrak{S}_n \colon \rho^2 = \mathrm{Id}, \forall i \in [n], \rho(i) \neq i \right\}.$$

Let us explain how to obtain a a combinatorial hypermap from an edge-labelled orientable hypermap $\mathfrak{m}$. By an edge-labelled map, we mean a map whose n edges are labelled in a bijective way by a chosen subset of the integers of size n, usually $[n]$.

Construction 2.4.10. *Let v be a vertex of $\mathfrak{m}$. Let $d(v)$ be the degree of v, i.e. the number of edges incident to it, and let $l_1, \ldots, l_{d(v)}$ be the labels of these edges, in counterclockwise order. We define the cyclic permutation*

$$\pi(v) = (l_1 \, \ldots \, l_{d(v)}).$$

We then define

$$\sigma(\mathfrak{m}) = \prod_{w \text{ white vertex}} \pi(w) \text{ and } \alpha(\mathfrak{m}) = \prod_{b \text{ black vertex}} \pi(b).$$

Let f be a face of $\mathfrak{m}$ of degree $d(f)$. Let $l_1, \ldots, l_{d(f)}$ the labels of the half-edges incident to f, in this order when exploring the face in the counterclockwise order. We set

$$\pi(f) = (l_1 \, \ldots \, l_{d(f)}),$$

and define

$$\varphi(\mathfrak{m}) = \prod_{f \text{ face}} \pi(f).$$

Notice that the permutations $\pi(w)$ for w a white vertex have disjoint supports. A similar remark holds for the black vertices and the faces. Note that the choice of exploring the vertices and faces in the counterclockwise direction is only a convention. Picking the opposite convention translates in the permutational model by replacing the permutations just defined by their inverses.

Lemma 2.4.11. *Let $\mathfrak{m}$ be a map with labelled half-edges. We have*

$$\sigma(\mathfrak{m})\alpha(\mathfrak{m})\varphi(\mathfrak{m}) = \mathrm{Id}.$$

A pictorial proof of this fact appears in Figure 2.2. A formal proof can be found in [LZ04].

A theorem of Edmonds then shows that Construction 2.4.10 gives a bijection between hypermaps and combinatorial maps. This result is also discussed in [LZ04, Chapter 1.3 and 1.4] and [MT01]. It shows that once the local structure of the map (and a labelling of the half-edges) is fixed, the map only depends on the underlying graph.

Theorem 2.4.12 ([Edm60]). *Let $n \geq 2$ be an even integer and $\sigma \in \mathfrak{S}_n$. Let $\mathfrak{C}(n, \sigma)$ be the set of hypermaps with n labelled edges $\mathfrak{m}$ such that $\sigma(\mathfrak{m}) = \sigma$. Then,*

$$\begin{aligned} \mathfrak{C}(n, \sigma) &\to \mathfrak{S}_n \\ \mathfrak{m} &\mapsto \alpha(\mathfrak{m}), \end{aligned}$$

is a bijection. When restricted to the subset of $\mathfrak{C}(n, \sigma)$ consisting of hypermaps with all their black vertices of degree 2, this mapping is a bijection to the set of involutions $\mathcal{I}_n$.

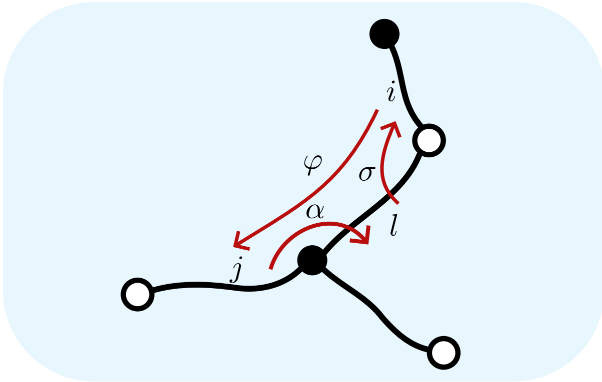


Figure 2.2: Proof by a drawing of Lemma 2.4.11: $\varphi(i) = j, \alpha(j) = l$, and $\sigma(l) = i$.

Considering maps with labelled half-edges is convenient, since specifying that automorphisms must preserve the labelling make the automorphism group trivial. However, it is sufficient to mark only one edge for the automorphism group to be trivial.

Definition 2.4.13 (Rooted map)**.** *A rooted map is a map with a marked half-edge.*

It is convenient to work with rooted maps when the possible labeling is not relevant. We can go from labelled maps to rooted maps by erasing all the labels, except for, say, the label 1.

Tutte's equation

To study generating series of planar maps, Tutte derived a quadratic equation based on the following decomposition procedure. Consider a rooted planar map $\mathfrak{m}$. It has a root vertex v_* and a root edge e_*. We contract e_* to produce a new map $\mathfrak{m}'$. There are two possible cases, illustrated in Figure 2.3.

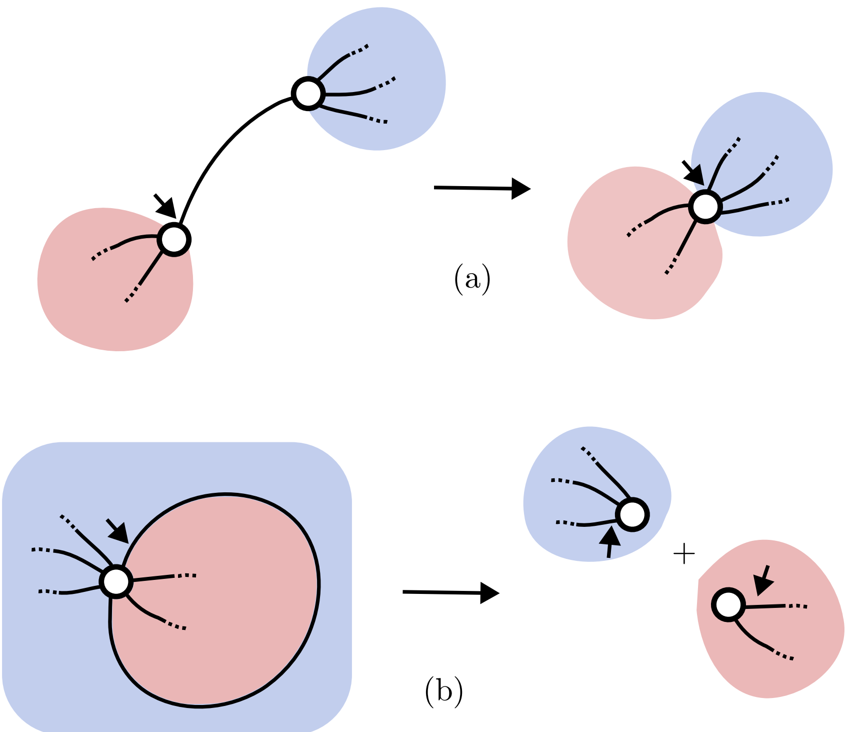


Figure 2.3: The two cases in Tutte's decomposition: (a) the contracted edge connects two distinct vertices, (b) the contracted edge is a loop. The root is marked by an arrow.

1. The root edge connects two distinct vertices v_* and v, respectively of degree d_* and d. When contracting e_*, we replace v_*, e_*, and v by a new root vertex v'_* of degree $d'_* = d_* + d - 2$. If the new map $\mathfrak{m}'$ has edges, the new root edge is the next edge after e_* in the clockwise direction around v if it exists, and the next edge around v_* otherwise.

2. The root edge is a self-loop (i.e. connects v_* to itself). When contracting e_* we cut the map into two connected components.

Denote by $m(n, d)$ the number of planar maps with n edges and root vertex of degree d. The decomposition just described allowed Tutte to derive an equation for the generating function of the $(m(n, d))_{n\geq 0, d\geq 1}$. Solving this equation allowed him to give a simple closed expression to the number of rooted planar maps with a given number of edges.

Note that we departed from the usual way this procedure is presented (in which the root edge is removed) by considering it in a dual way. This choice is made to match with the description of a Tutte-like equation described in Chapter 3, Section 3.5.

2.4.2 Coverings and Hurwitz numbers

Hurwitz numbers were introduced by Hurwitz in the XIXth century [Hur91]. They are numbers of ramified coverings of two-dimensional surfaces, up to isomorphism. In this section, we give first the topological picture, and then define them in term of number of some factorization of permutations. We may also understand them in term of particular maps. This discussion prepares the introduction of maps of unitary type, introduced in Chapter 3. See the review of Lando [Lan11] for more on Hurwitz numbers.

A covering is a surjective open map that is locally a homeomorphism. A ramified covering $\pi\colon \hat{S} \to S$ between two compact topological spaces is a mapping such that upon removing a finite set of *ramification points* $\hat{R}$ from $\hat{S}$, the mapping $\pi|_{\hat{S}\setminus\hat{R}}\colon \hat{S}\setminus\hat{R} \to S\setminus\pi(\hat{R})$ is a covering. We set for convenience $R = \pi(\hat{R})$, the set of *branch points*.

Let us assume that $\hat{S}$ and S are compact Riemann surfaces. It is indeed a natural framework: any holomorphic mapping $\pi\colon \hat{S} \to S$ may be seen as a branched covering, as we now explain. For any point $p \in \hat{S}$, π may be expressed in a chart near p as

$$z \mapsto z^{k(p)}$$

with $k(p) \geq 1$ an integer. This is the degree of π at p. We define the set

$$\hat{R} = \left\{p \in \hat{S}\colon k(p) > 1\right\}.$$

Then, π is a ramified covering with ramification points $\hat{R}$.

Given $q \in S$, write $\pi^{-1}(\{q\}) = \{p_1, \dots, p_m\}$. The integer

$$d = \sum_{i=1}^{m} k(p_i)$$

does not depend on q, it is the *degree* of the covering π. The integers $k(p_1), \dots, k(p_m)$ form a partition of d, called the ramification profile at q.

We now explain how an isomorphism class of ramified covering may be encoded in terms of permutations. A detailed explanation of this may be found in the book of Miranda [Mir97]. Fix a point $q \in S \setminus R$. The fundamental group $\pi_1(S \setminus R)$ acts on the fiber above q, $\pi^{-1}(\{q\})$, as follows. Pick $p \in \pi^{-1}(\{q\})$, $[\gamma] \in \pi_1(S \setminus R)$, and γ a representative of $[\gamma]$ based at q. We can lift γ by π to a curve $\hat{\gamma}$ on $\hat{S} \setminus \hat{R}$ that starts at p. The other endpoint of $\hat{\gamma}$, p', does not depend on the choice of representative in $[\gamma]$. It is the image of p. This construction gives a morphism

$$\pi_1(S \setminus R) \to \mathfrak{S}(\pi^{-1}(\{q\})).$$

If we label the d preimages of q, we obtain a morphism

$$\rho\colon \pi_1(S \setminus R) \to \mathfrak{S}_d.$$

We call this mapping a *monodromy representation* of π. Note that there is one monodromy representation for each way to label the preimage of q, any two such monodromy representations are related by a conjugation by a permutation in $\mathfrak{S}_d$. The image of ρ is a subgroup of $\mathfrak{S}_d$ which acts transitively on $[d]$ as $\hat{S}$ is connected. In fact, the monodromy representation characterizes the isomorphism class of ramified coverings, or equivalently of holomorphic maps.

Proposition 2.4.14 ([Mir97, Proposition 4.9])**.** *There is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{isomorphism class of holomorphic maps}\\ \pi\colon \hat{S} \to S\\ \textit{of degree } d \textit{ with branch points in } R\end{matrix}\right\} \to \left\{\begin{matrix}\textit{group morphisms } \pi_1(S\setminus R) \to \mathfrak{S}_d\\ \textit{whose image is a transitive group}\\ \textit{(up to conjugacy in } \mathfrak{S}_d\textit{)}\end{matrix}\right\}.$$

Assume now that $S = \mathbb{CP}^1$, and write $R = \{q_1, \dots, q_m\}$. For all $i \in [m]$, let γ_i be a small loop in the positive direction enclosing q_i only, and denote by $[\gamma_i]$ its class in $\pi(\mathbb{CP}^1 \setminus R)$. The data of the m permutations

$$\sigma_1 = \rho([\gamma_1]), \dots, \sigma_m = \rho([\gamma_m])$$

determines the isomorphism class of π. Note that

$$\sigma_1 \cdots \sigma_m = \mathrm{Id}\,.$$

The bottom line is that, as a corollary of Proposition 2.4.14, there is a one-to-one mapping

$$\left\{\begin{matrix}\text{isomorphism class of holomorphic maps}\\ \pi\colon \hat{S} \to \mathbb{CP}^1\\ \text{of degree } d \text{ with branch points in } R\end{matrix}\right\} \to \left\{\begin{matrix}\text{conjugacy classes of}\\ (\sigma_1, \dots, \sigma_m) \in \mathfrak{S}_d^m \text{ with } \sigma_1 \cdots \sigma_m = \mathrm{Id}\\ \text{such that } \langle \sigma_1, \dots, \sigma_m \rangle \text{ is transitive}\end{matrix}\right\}.$$

We now give a few applications of this fact. The main point of what follows is that a map $\mathfrak{m}$ on S can be lifted to a map $\hat{\mathfrak{m}}$ on $\hat{S}$. If we fix $\mathfrak{m}$ and vary the covering, we obtain a family of maps.

Maps as coverings

Consider a ramified covering of the Riemann sphere $\pi\colon \hat{S} \to \mathbb{CP}^1$ with three branch points: $0, 1$, and ∞. Let σ_0, σ_1, and σ_∞ the three permutations describing the monodromy above the three branch points. It turns out this data describes a hypermap. This fact is discussed in [LZ04, Construction 1.3.22]. To see this, assume that $e\colon [0,1] \to \mathbb{CP}^1$ is a smooth path, say a straight line in $\mathbb{C} \subset \mathbb{CP}^1$, with $e(0) = 0$ and $e(1) = 1$. This defines an embedded graph, and thus a map $\mathfrak{m}$, with one edge. We now lift e to a $\hat{S}$. This gives a graph embedding: the white vertices are the preimages of 0, the black vertices are the preimages of 1, the edges are the preimages of e, and the faces are the preimages of $\mathbb{CP}^1 \setminus e([0,1])$ – which are discs around ∞. In particular, the preimages of $\mathbb{CP}^1 \setminus e([0,1])$ are homeomorphic to discs. See Figure 2.4 In this way, an equivalence class of ramified covering to the sphere corresponds to a (orientable) map $\hat{\mathfrak{m}}$.

The edges are naturally labelled by the choice of labelling of preimages of the fixed point q. We have

$$\sigma_{\hat{\mathfrak{m}}} = \sigma_0, \alpha_{\hat{\mathfrak{m}}} = \sigma_1, \varphi_{\hat{\mathfrak{m}}} = \sigma_\infty.$$

Note that the factorization property $\sigma_0\sigma_1\sigma_\infty = \mathrm{Id}$ corresponds to Lemma 2.4.11.

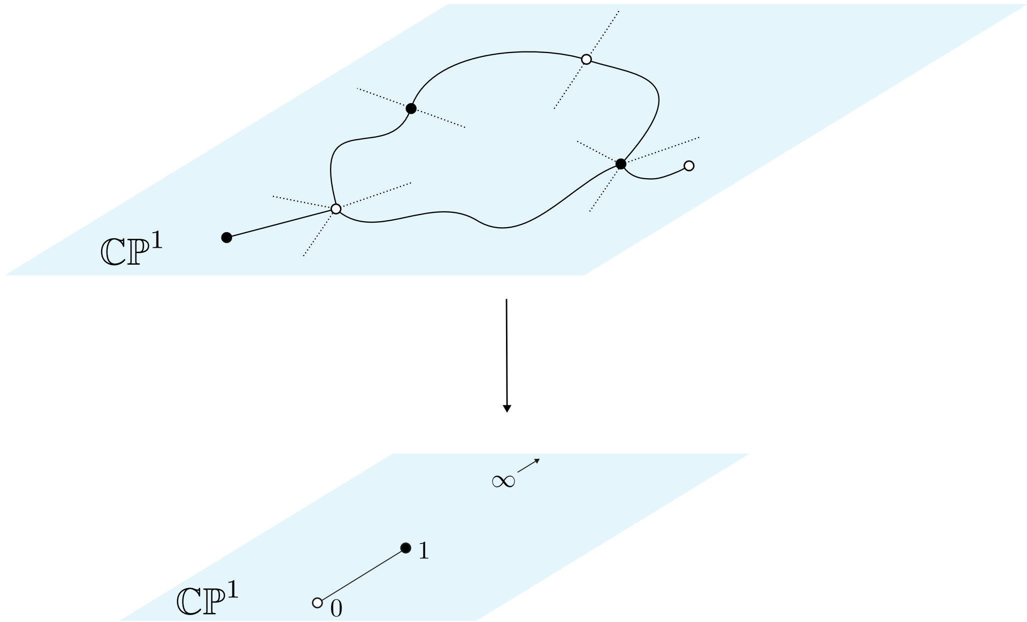


Figure 2.4: A planar map (upper sheet) as the lift by a ramified covering of an edge on the lower sheet. The ramification profile of the white vertex is $\lambda_\circ = (3^1 2^1 1^1)$ and of the black vertex is $\lambda_\bullet = (3^1 2^1 1^1)$.

Hurwitz numbers

Hurwitz numbers are numbers (up to isomorphism) of branched coverings with l points with specified ramification profiles. We give a combinatorial definition.

Definition 2.4.15 (Hurwitz numbers)**.** *Let $d, l \geq 1$ be integers, and $\lambda_1, \ldots, \lambda_l \vdash d$. We define the general Hurwitz numbers as*

$$h(\lambda_1, \ldots, \lambda_l) = \frac{1}{d!} \# \left\{ (\sigma_1, \ldots, \sigma_l) \in \mathfrak{S}_d^l : \begin{array}{l} \bullet \quad \text{for all } i \in [l], \sigma_i \in \mathcal{C}_{\lambda_i} \\ \bullet \quad \sigma_r \sigma_{r-1} \cdots \sigma_1 = \mathrm{Id} \\ \bullet \quad \langle \sigma_1, \ldots \sigma_r \rangle \text{ is transitive} \end{array} \right\}.$$

Let $r \geq 0$, and λ, μ, ν. The triple Hurwitz numbers are

$$h^r(\lambda, \mu, \nu) = h(\lambda, \mu, \nu, \underbrace{(2^1 1^{d-1}), \ldots, (2^1 1^{d-1})}_{r \text{ times}}).$$

We now explain how the triple Hurwitz numbers may be seen as number of maps of a special kind, which we now define. The construction that follows is inspired from [Joh12]. Fix $r, d \geq 1$, $\lambda, \mu, \nu \vdash d$, and set $\xi_k = \exp(2i\pi k/(r+1))$ for $k \in \{0, 1, \ldots, r\}$. Consider a ramified covering π with set of branch points

$$R = \{0, 1 = \xi_0, \xi_1, \ldots, \xi_r, \infty\}.$$

With $0, 1, \infty$ of ramification profile λ, μ, ν respectively, and for $k \in [r]$, ξ_k of ramification profile $(2^1 1^{d-1})$. Let C be the unit circle in $\mathbb{C} \subset \mathbb{CP}^1$, and C_k the arc between ξ_k and ξ_{k+1} (with the convention that $\xi_{r+1} = 1$). The ξ_k's and the C_k's define an embedded graph with two faces, $r+1$ edges, and $r+1$ vertices. This defines a map $\mathfrak{m}$.

The preimage by π of the C_k's and of the branch points ξ_k for $k \in \{0, \ldots, r\}$ defines an embedded graph on $\hat{S}$, and hence a map $\hat{\mathfrak{m}}$. There are two types of vertices: preimages of 1 – say white vertices – and preimages of $\xi_k, k \in [r]$ – say black vertices. There are also two types of faces: preimages of the lower hemisphere (containing 0) and preimages of the upper hemisphere (containing ∞). We may orient C in the positive orientation. The preimages of the C_k's are then oriented edges. The black vertices are of degree either 4 or 2. By following an edge we go from a preimage of ξ_{k-1} for some $k \in [r+1]$ to a preimage of ξ_k. See Figure 2.5.

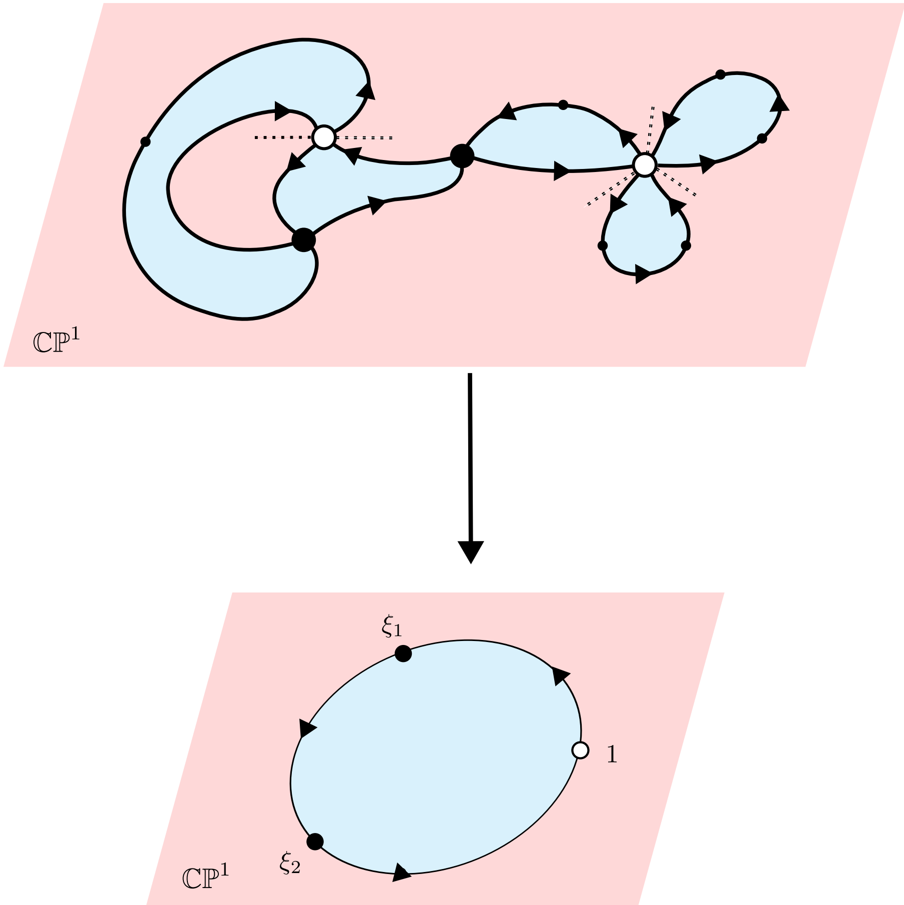


Figure 2.5: Example of covering counted by a triple Hurwitz number. Here, $r = 2$ and the ramification profiles are $\lambda = (1^5), \mu = (3^1 2^1), \nu = (4^1 1^1)$.

Remark 2.4.16. Note that by the Riemann-Hurwitz theorem, we can deduce the genus of the resulting map – or of the surface $\hat{S}$ – from r, d, λ, μ, ν. Here, we can use it in the form of Euler's formula. There are $r(d-1) + l(\mu)$ vertices, $(r+1)d$ edges, and $l(\lambda) + l(\nu)$ faces. Hence,

$$2 - 2g = (r(d-1) + l(\mu)) - (r+1)\,d + (l(\lambda) + l(\nu)) = l(\lambda) + l(\mu) + l(\nu) - d - r.$$

With this point of view, the class of π is determined by $\hat{\mathfrak{m}}$. In a sense that we do not make precise at this point, the triple Hurwitz number count maps such as $\hat{\mathfrak{m}}$. Indeed, in Chapter 3, we will consider *maps of unitary type* who can be seen as generalization of the maps constructed by lifting $\mathfrak{m}$.

A further refinement of triple Hurwitz number, relevant for the integrals over the unitary group, is the notion of *monotone* Hurwitz numbers.

Definition 2.4.17 (Monotone Hurwitz numbers)**.** *Let $d, l \geq 1$ be integers, and $\lambda, \mu, n \vdash d$. We define the triple monotone Hurwitz numbers as*

$$\vec{h}^{\,r}(\lambda,\mu,\nu) = \frac{1}{d!}\#\left\{ (\sigma,\gamma,\rho,\tau_1,\ldots,\tau_r) \in \mathfrak{S}_d^{r+3} : \begin{array}{l} \bullet\ \ \sigma \in \mathcal{C}_\lambda, \gamma \in \mathcal{C}_\mu, \rho \in \mathcal{C}_\nu \\ \bullet\ \ \rho\tau_r\tau_{r-1}\cdots\tau_1\gamma\sigma = \mathrm{Id} \\ \bullet\ \ \langle \sigma,\gamma,\rho,\tau_1,\ldots\tau_r\rangle \text{ is transitive} \\ \bullet\ \ \forall i \in [r], \tau_i = (a_i\, b_i) \text{ with } a_i < b_i \\ \bullet\ \ b_1 \leq b_2 \leq \cdots \leq b_r \end{array} \right\}.$$

Simple and double monotone Hurwitz numbers are defined respectively by

$$\vec{h}^{\,r}(\lambda) = \vec{h}^{\,r}(\lambda,(1^d),(1^d)) \quad \text{and} \quad \vec{h}^{\,r}(\lambda,\mu) = \vec{h}^{\,r}(\lambda,(1^d),\mu).$$

Actually, in Chapter 3, we will use a different version of triple monotone Hurwitz number, where we fix the permutations σ, γ, ρ instead of talking them arbitrary in a class. This corresponds to making a choice of labelling of the sheets of the covering.

We may also consider covering of higher genus compact surfaces. We defer this discussion to Appendix 8.2.

2.5 Computation of moments: direct approach

We now explain how moments may be computed directly. The most important case is the one of the GUE.

2.5.1 The Wick calculus and maps

Let us explain how the moments of the GUE and the enumeration of orientable maps are related. An introduction to this can also be found in the review [Zvo97] by Zvonkin. This idea is based on the following formula, due to Isserlis [Iss18] and rediscovered later by Wick [Wic50]. It can be seen as a consequence of the moment-cumulant relations (2.3).

Proposition 2.5.1 (Isserlis theorem or Wick formula). *Let $(X_1, \ldots, X_n)$ be a centered normal random vector. We have*

$$\mathbb{E}\left[X_1 \cdots X_n\right] = \sum_{\alpha \in \mathcal{I}_n} \prod_{(i\,j) \in \alpha} \mathbb{E}\left[X_i X_j\right].$$

Proof. The key property of Gaussian random vectors that we use is that the only joint cumulants of the $X_1, \ldots, X_n$ that can be nonzero are the second ones: for $l \geq 1$ and $i \colon [l] \to [n]$,

$$\kappa_l(X_{i(1)}, \ldots, X_{i(l)}) = \delta_{l,2} \kappa(X_{i(1)} X_{i(2)}).$$

The moment-cumulant relations (2.3) immediately give:

$$\mathbb{E}\left[X_1 \cdots X_n\right] = \sum_{\Pi \in \mathcal{P}_n} \prod_{B \in \Pi} \kappa_{\#B}(X_p; p \in B) = \sum_{\substack{\Pi \in \mathcal{P}_n \\ \#\Pi = 2}} \prod_{B \in \Pi} \kappa_{\#B}(X_p; p \in B).$$

The result is a consequence of the fact that the mapping from $\mathcal{I}_n$ to the set of partitions of $[n]$ with blocks of size 2, $\alpha \mapsto 0_\alpha$ is a bijection. □

The moments of the GUE are simply expectations of product of traces of powers of a GUE matrix X: the computation of these moments is thus based on repeated use of Proposition 2.5.1. We now explain how the computation goes with an additional twist. Firstly, we compute cumulants, they correspond to connected objects, maps, while the moments correspond to disconnected maps. Secondly, we compute the cumulants not of X but of AX for A a deterministic matrix. In the physics literature, A corresponds to an "external field", see for instance [BH16]. We will see below that this has a combinatorial interest.

Proposition 2.5.2. *Let X be a GUE matrix of size $N \times N$, and A be a deterministic square matrix of size $N \times N$. Let $n \geq 1$ be an integer and let $\lambda \vdash n$ be a partition of n. We have*

$$\kappa_{l(\lambda)}\left[\operatorname{Tr}\left((AX)^{\lambda_1}\right), \ldots, \operatorname{Tr}\left((AX)^{\lambda_1}\right)\right] = \frac{(n-1)!}{\#\mathcal{C}_\lambda} N^{2-l(\lambda)} \sum_{\nu \vdash n} \sum_{\substack{\mathfrak{m} \text{ is a rooted map with} \\ \text{vertex profile } \lambda \text{ and face profile } \nu}} N^{-2g_\mathfrak{m}} \operatorname{tr}_\nu(A),$$

where $g_\mathfrak{m}$ is the genus of a map $\mathfrak{m}$.

Proof. We start by fixing a permutation $\sigma \in \mathcal{C}_\lambda$. We compute first the moments:

$$\mathbb{E}\left[\mathrm{Tr}_\lambda(AX)\right] = \sum_{\substack{i\colon [n]\to[N] \\ j\colon [n]\to[N]}} \left(\prod_{k=1}^{n} A_{i(k),j(k)}\right) \mathbb{E}\left[\prod_{k=1}^{n} X_{j(k),i(\sigma(k))}\right].$$

We use Wick's formula, Proposition 2.5.1, to compute the expectation. Notice that the covariance of element of the Gaussian vector $(X_{i,j})_{i,j\in[N]}$ are

$$\mathbb{E}\left[X_{ij}X_{kl}\right] = \frac{\delta_{i,l}\delta_{j,k}}{N} \text{ for } i,j,k,l \in [N].$$

We obtain

$$\mathbb{E}\left[\mathrm{Tr}_\lambda(AX)\right] = \sum_{\substack{i\colon [n]\to[N] \\ j\colon [n]\to[N]}} \left(\prod_{k=1}^{n} A_{i(k),j(k)}\right) \sum_{\alpha\in\mathcal{I}_n} \prod_{(p\,q)\in\mathrm{Cycles}(\alpha)} \frac{\delta_{j(p),i(\sigma(q))}\delta_{i(\sigma(p)),j(q)}}{N}.$$

Notice that after exchanging the sum on i,j and the sum on α, the delta functions can be rewritten as $\delta_{j,i\sigma\alpha^{-1}} = \prod_{k=1}^{n} \delta_{j(k),i\sigma\alpha^{-1}(k)}$. This gives

$$\mathbb{E}\left[\mathrm{Tr}_\lambda(AX)\right] = \sum_{\alpha\in\mathcal{I}_n} \sum_{i\colon [n]\to[N]} \left(\prod_{k=1}^{n} A_{i(k),i\sigma\alpha^{-1}(k)}\right) \frac{1}{N^{n/2}} = \sum_{\alpha\in\mathcal{I}_n} \mathrm{Tr}_{\sigma\alpha^{-1}}(A^{\otimes n})\frac{1}{N^{n/2}}.$$

We notice that the sum on (σ,α) is a sum on possibly disconnected combinatorial maps. If we decompose on the orbits of the group $\langle\sigma,\alpha\rangle$, we have following the discussion in Section 2.2.2:

$$\kappa_{l(\lambda)}\left(\mathrm{Tr}\left((AX)^{\lambda_1}\right),\ldots,\mathrm{Tr}\left((AX)^{\lambda_{l(\lambda)}}\right)\right) = \sum_{\substack{\alpha\in\mathcal{I}_n \\ \langle\sigma,\alpha\rangle \text{ acts transitively on } [n]}} \mathrm{Tr}_{\sigma\alpha^{-1}}(A^{\otimes n})\frac{1}{N^{n/2}}.$$

We recognize a sum over maps: (σ,α) is a connected combinatorial map, and Theorem 2.4.12 implies that we can replace the sum over α with a sum over half-edge labelled maps with vertex profile determined by σ. We have

$$\begin{aligned}\kappa_{l(\lambda)}\left(\mathrm{Tr}\left((AX)^{\lambda_1}\right),\ldots,\mathrm{Tr}\left((AX)^{\lambda_{l(\lambda)}}\right)\right) &= \sum_{\substack{\mathfrak{m} \text{ is a labelled map} \\ \sigma_\mathfrak{m}=\sigma}} \mathrm{Tr}_{\varphi_\mathfrak{m}^{-1}}(A^{\otimes n})\frac{1}{N^{n/2}} \\ &= \sum_{\substack{\mathfrak{m} \text{ is a labelled map} \\ \sigma_\mathfrak{m}=\sigma}} \mathrm{tr}_{\varphi_\mathfrak{m}^{-1}}(A^{\otimes n})N^{\#\varphi_\mathfrak{m}-n/2}.\end{aligned}$$

Euler's formula – see Theorem 2.4.4 – gives

$$\#\sigma_\mathfrak{m} - \frac{n}{2} + \#\varphi_\mathfrak{m} = 2 - 2g_\mathfrak{m}.$$

The overall power of N is then $\#\varphi_\mathfrak{m} - n/2 = 2 - 2g - \#\sigma_\mathfrak{m} = 2 - 2g_\mathfrak{m} - l(\lambda)$. To go from half-edge labelled maps to rooted maps, we average over all possible choices of $\sigma \in \mathcal{C}_\lambda$ and sum on the possible cyclic types of $\varphi_\mathfrak{m}$:

$$\begin{aligned}&\kappa_{l(\lambda)}\left(\mathrm{Tr}\left((AX)^{\lambda_1}\right),\ldots,\mathrm{Tr}\left((AX)^{\lambda_{l(\lambda)}}\right)\right) \\ &= N^{2-2l(\lambda)}\frac{1}{\#\mathcal{C}_\lambda}\sum_{\sigma\in\mathcal{C}_\lambda}\sum_{\substack{\mathfrak{m} \text{ is a labelled map} \\ \sigma_\mathfrak{m}=\sigma}} \mathrm{tr}_{\varphi_\mathfrak{m}^{-1}}(A^{\otimes n})N^{-2g_\mathfrak{m}} \\ &= N^{2-2l(\lambda)}\frac{1}{\#\mathcal{C}_\lambda}\sum_{\nu\vdash n}\sum_{\substack{\mathfrak{m} \text{ is a labelled map with} \\ \text{vertex profile } \lambda \text{ and face profile } \nu}} \mathrm{tr}_\nu(A)N^{-2g_\mathfrak{m}}.\end{aligned}$$

Each rooted map correspond to $(n-1)!$ half-edge labelled maps with root labelled by 1. We obtain the result. $\square$

In particular, if we choose A to be the diagonal matrix with formal coefficients $z_1, \ldots, z_N$ on the diagonal, the quantity

$$\kappa_{l(\lambda)}\left(\bigotimes_{i=1}^{l(\lambda)} \operatorname{Tr}\left((AX)^{\lambda_i}\right)\right) \tag{2.14}$$

is a symmetric polynomial in the variables $\boldsymbol{z} = (z_i)_{i\in[N]}$. A basis of symmetric polynomials is provided by the power-sum polynomials, defined by

$$p_\nu = \prod_{j=1}^{l(\nu)} p_{(\nu_j)} \quad \text{where} \quad p_{(k)}(x_1, \ldots, x_N) = \sum_{i=1}^{N} x_i^k, \text{ for } k \geq 1. \tag{2.15}$$

Proposition 2.5.2 expresses the polynomial (2.14) it in the basis of the power-sum polynomials $p_\mu(\boldsymbol{z})$ as

$$\kappa_{l(\lambda)}\Big(\bigotimes_{i=1}^{l(\lambda)} \operatorname{Tr}\left((AX)^{\lambda_i}\right)\Big)$$
$$= \sum_{g\geq 0}\sum_{\nu\vdash n} \frac{(n-1)!}{\#\mathcal{C}_\lambda} N^{2-l(\lambda)-2g} \# \left\{ \mathfrak{m}\colon \begin{array}{c} \mathfrak{m} \text{ is an orientable rooted} \\ \text{map of genus } g \text{ with vertex} \\ \text{profile } \lambda \text{ and face profile } \nu \end{array} \right\} p_\nu(\boldsymbol{z}).$$

One is then tempted to use this computation to express the free energy of a unitarily-invariant Hermitian one-matrix model: given a potential $V(x) = \sum_{k\geq 1} \frac{t_k}{k} x^k$, we have formally

$$\begin{aligned}
\frac{1}{N^2}\ln\frac{Z^N_{2,V}}{Z^N_{2,0}} &= \sum_{n\geq 1}\frac{N^{n-2}}{n!}\sum_{i\colon [n]\to\mathbf{N}^*}\left(\prod_{p=1}^{n}\frac{t_{i(p)}}{i(p)}\right)\kappa_n\big(\bigotimes_{p=1}^{n}\operatorname{Tr} X^{i(p)}\big) \\
&= \sum_{n\geq 1}\frac{N^{n-2}}{n!}\sum_{\lambda \text{ with } l(\lambda)=n}\#\mathcal{C}_\lambda\left(\prod_{p=1}^{n} t_{\lambda_p}\right)\kappa_n\big(\bigotimes_{p=1}^{n}\operatorname{Tr} X^{\lambda_p}\big) \\
&= \sum_{n\geq 1}\frac{1}{n}\sum_{\lambda \text{ with } l(\lambda)=n}\left(\prod_{p=1}^{n} t_{\lambda_p}\right)\left(\sum_{\substack{\mathfrak{m} \text{ is an orientable rooted map with} \\ \text{vertex profile } \lambda}} N^{-2g}\right) \\
&= \sum_{g\geq 0}\frac{1}{N^{2g}}\left(\sum_{\lambda}\frac{1}{l(\lambda)}\left(\prod_{p=1}^{l(\lambda)} t_{\lambda_p}\right)\#\left\{\mathfrak{m}\colon \begin{array}{c}\mathfrak{m} \text{ is an orientable rooted} \\ \text{map of genus } g \\ \text{with vertex profile } \lambda\end{array}\right\}\right),
\end{aligned} \tag{2.16}$$

where in the last line, the sum on λ is on all partition of a positive integer. The first equality is consequence of the fact that $\ln\frac{Z^N_{2,V}}{Z^N_{2,0}}$ is a cumulant generating function. The second equality is a consequence of the orbit-stabilizer theorem for the action of $\mathfrak{S}_n$ on functions $[n] \to \mathbf{N}^*$, acting by composition on the right. In the course of this computation, we exchanged several infinite sums. This is not justified in general: the above equality only holds in the sense of formal power series, see [Eyn11] for instance. By this, we mean that the derivatives with respect to any number of t_i's evaluated at $t_j = 0$ for $j \geq 1$ of the left and right sides of the equation above coincide. In fact, the integral on the left side or/and the series of maps on the right side may not converge in general. The expansion (2.16) is sometimes called a formal matrix integral.

Remark 2.5.3. This formal computation generalizes to the case of $m \geq 1$ GUE matrices. The maps thus obtained are maps with each edge colored in one of m colors.

Taking for instance the potential for the Ising model on a random lattice:

$$V(X, Y) = gX^4 + gY^4 - cXY,$$

the formal integral is a power series in g and c. The coefficient of $g^k(-c)^l N^{-2g}$ is (up to symmetry factors) the number of maps:

- of genus g;
- with edges that are of one of two colors, say red or blue;
- with k vertices of degree 4, each of them incident to all red or all blue edges;
- with l vertices of degree 2 incident to one blue edge and one red edge, the frustrated edges of the Ising model.

Remark 2.5.4. Up to now, we only discussed orientable maps. Cumulants of the GOE ($\beta = 1$) and the GSE ($\beta = 4$) can also be expressed in terms of maps: it can also be shown using Wick calculus. However, rather than only orientable maps, maps of all possible topologies – including maps on non-orientable surfaces – are counted. This fact was shown in the physics litterature by Cicuta [Cic82]. A random matrix-centered derivation can be found in the work of Mulase and Waldron [MW03].

2.5.2 Moments of the β-ensembles and Jack polynomials

We saw in Section 2.5.1 that when $\beta = 2$, we could compute the joint moments and cumulants of the GUE in terms of maps. We now consider the computation of the joint moments of the β-ensemble for any $\beta > 0$. A priori, for a general β, the moments are given by

$$\langle p_\nu(\lambda_1, \dots, \lambda_N) \rangle^N_{\beta,0},$$

with ν the partition of a positive integer, and p_ν the power-sum polynomial defined in (2.15).

It turns out that the most natural polynomials to work with the β-ensemble are the Jack polynomials. They were introduced by Jack in [Jac70] to study zonal polynomials. Zonal and Schur polynomials appear naturally when considering integrals over the orthogonal or unitary group respectively – corresponding to $\beta = 1$ and $\beta = 2$. The interpolation between the measures $\nu^N_{1,0}$ and $\nu^N_{2,0}$ is mirrored in the interpolation between zonal and Schur polynomials provided by Jack polynomials. Let us now define the Jack polynomial and give some of their properties. Two references are the review of Stanley [Sta89] or the book of Macdonald [Mac15]. Recall that the monomial symmetric polynomials are defined for a partition μ by

$$m_\mu(x_1, \dots, x_k) = \sum_{\sigma \in \mathfrak{S}_k} x^{\mu_1}_{\sigma(1)} x^{\mu_2}_{\sigma(2)} \cdots x^{\mu_k}_{\sigma(k)},$$

with $\mu_i = 0$ if $i > l(\mu)$. They form a basis of the symmetric polynomials. Given a symmetric polynomial P, we denote by $[m_\mu]P$ the coefficient of m_μ when P is expressed in the basis of the monomial symmetric polynomials.

Definition 2.5.5 (Jack polynomials)**.** *Let α be a real parameter. We define the inner product $\langle \cdot, \cdot \rangle_\alpha$ on the space of symmetric polynomials by the orthogonality relations:*

$$\langle p_\mu, p_\nu \rangle_\alpha = \frac{|\mu|!}{\# \mathcal{C}_\mu} \alpha^{l(\mu)} \delta_{\mu,\nu},$$

for μ, ν two partitions of integers. The Jack polynomials $(J_\mu(\cdot; \alpha))$ form a basis for the set of homogeneous symmetric polynomials. They are determined by:

- *If $\mu \neq \nu$ are two partitions, $\langle J_\mu, J_\nu \rangle_\alpha = 0$;*
- *We have $[m_\mu]J_\nu = 0$ unless $\mu \preceq \nu$;*
- *Let μ be a partition of an integer k, $[x_1 \cdots x_k]J_\mu = k!$.*

In particular, when $\alpha = 1$ the functions $J_\mu(\cdot; 1)$ are (up to a constant) the Schur polynomials, and when $\alpha = 2$ the function $J_\mu(\cdot; 2)$ are the zonal polynomials. A formula about Jack polynomials we will use in the sequel is Cauchy's formula.

Theorem 2.5.6 (Cauchy Theorem for Jack polynomials [Sta89, Proposition 2.1])**.** *Let $\alpha \in \mathbb{R}$, and $\boldsymbol{x} = (x_i)_{i\geq 1}$ and $\boldsymbol{y} = (y_i)_{i\geq 1}$ be two families of formal variables. We have*

$$\prod_{i,j\geq 1} \frac{1}{(1 - x_i y_j)^{1/\alpha}} = \sum_\mu \frac{J_\mu(\boldsymbol{x}; \alpha) J_\mu(\boldsymbol{y}; \alpha)}{\langle J_\mu, J_\mu \rangle_\alpha},$$

where the sum is on all partitions of integers (including the empty one).

Averages under the β-ensemble measure may be seen as a degeneration of Selberg's integrals. Selberg's integral is

$$\int_{[0,1]^N} |\Delta(\boldsymbol{\lambda})|^\beta \prod_{j=1}^N \lambda_j^x (1 - \lambda_j)^y \mathrm{d}\boldsymbol{\lambda} = \prod_{j=0}^{N-1} \frac{\Gamma(1 + (j+1)\frac{\beta}{2})}{\Gamma(1 + \frac{\beta}{2})} \frac{\Gamma(x + j\frac{\beta}{2})\Gamma(y + j\frac{\beta}{2})}{\Gamma(x + y + (N - 1 - j)\frac{\beta}{2})}, \tag{2.17}$$

for $x, y, \beta > 0$. It can be seen as the partition function of a multi-particle generalization of the Beta distribution. It was shown by Kadell [Kad97] that expectation values of Jack polynomials against the integrand of the Selberg integral could be exactly computed for even parameter β.

The partition function of the β-ensemble may be computed using the Selberg-Mehta integral, which can be seen as a degeneration of Selberg's integral (see the book of Mehta [Meh04, Section 17.6]). By doing appropriate changes of variables, we obtain the Selberg-Mehta integral:

$$\mathcal{Z}_{\beta,0}^N = \int_{\mathbb{R}^N} |\Delta(\boldsymbol{\lambda})|^\beta \mathrm{e}^{-\frac{\beta}{4} N \sum_{i=1}^N \lambda_i^2} \mathrm{d}\boldsymbol{\lambda} = (2\pi)^{N/2} \left(\frac{\beta}{2} N\right)^{-N(\beta/2(N-1)+1)/2} \prod_{j=1}^N \frac{\Gamma(1 + j\frac{\beta}{2})}{\Gamma(1 + \frac{\beta}{2})}. \tag{2.18}$$

Rather than the power-sum symmetric polynomials, it seems more natural to compute the averages under the β-ensemble of the Jack polynomials. Indeed, we have the following Theorem, which was a conjecture of Goulden and Jackson [GJ97] before being proven by Okounkov [Oko97].

Theorem 2.5.7. *Let $N \geq 1$ be an integer and $\beta > 0$. We have for all partition μ of an integer m that*

$$\nu_{\beta,0}^N \left[J_\mu(\boldsymbol{\lambda}; \frac{2}{\beta}) \right] = \begin{cases} N^{-m/2} J_\mu(1_N; \frac{2}{\beta}) [p_2^{m/2}] J_\mu(\cdot; \frac{2}{\beta}), & \text{if } m \text{ is even}, \\ 0 & \text{if } m \text{ is odd}, \end{cases}$$

where $1_N = (\underbrace{1, \ldots, 1}_{N \text{ times}})$.

We may relate the quantities $\nu_{\beta,0}^N \left[J_\mu(\boldsymbol{\lambda}; \frac{2}{\beta}) \right]$ and $\nu_{\beta,0}^N \left[p_\mu(\boldsymbol{\lambda}) \right]$ by expressing J_μ in the basis of power-sum polynomials. In particular, we have [Sta89, Proposition 2.2]:

$$J_{(n)}(\boldsymbol{\lambda}; \frac{2}{\beta}) = \sum_{\mu \vdash n} \left(\frac{2}{\beta}\right)^{n - l(\mu)} \frac{n!}{\prod_{j\geq 1} j^{m_j^\mu} m_j^\mu!} p_\mu,$$

where m_j^μ is the number of parts of μ of size $j \in \mathbf{N}^*$. While possible in principle, relating the two bases of symmetric polynomials to compute the moments of the β-ensemble is arduous in practice.

It was noticed by Goulden, Harer, and Jackson [GHJ01, Proposition 5.1] that the free energy of the β-ensemble could be expressed formally as a series of Jack polynomials. Indeed, we have at the level of formal power series

$$\begin{aligned}\nu_{\beta,0}^N\left[\exp\left(\frac{\beta}{2}\sum_{j\geq 1}\frac{z^{j/2}}{j}p_j(\boldsymbol{y})p_j(\boldsymbol{\lambda})\right)\right] &= \nu_{\beta,0}^N\left[\exp\left(\frac{\beta}{2}\sum_{j,k,l\geq 1}\frac{z^{j/2}y_k^j\lambda_l^j}{j}\right)\right]\\ &= \nu_{\beta,0}^N\left[\exp\left(-\frac{\beta}{2}\sum_{k,l\geq 1}\ln\Big(1-z^{1/2}y_k\lambda_l\Big)\right)\right]\\ &= \nu_{\beta,0}^N\left[\prod_{k\geq 1}\prod_{l=1}^N(1-y_k\lambda_l)^{-\beta/2}\right]\\ &= \nu_{\beta,0}^N\left[\sum_{\mu} z^{|\mu|/2}\frac{J_\mu(\boldsymbol{y};\frac{2}{\beta})J_\mu(\boldsymbol{\lambda};\frac{2}{\beta})}{\langle J_\mu,J_\mu\rangle_{2/\beta}}\right],\end{aligned}$$

where in the last line we used Theorem 2.5.6. Theorem 2.5.7 then implies

$$\nu_{\beta,0}^N\left[\mathrm{e}^{\frac{\beta}{2}\sum_{j\geq 1}\frac{z^{j/2}}{j}p_j(\boldsymbol{y})p_j(\boldsymbol{\lambda})}\right] = \sum_{\substack{\mu\\ |\mu|\text{ even}}}\left(\frac{z}{N}\right)^{|\mu|/2}\frac{J_\mu(\boldsymbol{y};\frac{2}{\beta})J_\mu(1_N;\frac{2}{\beta})}{\langle J_\mu,J_\mu\rangle_{2/\beta}}[p_2^{|\mu|/2}]J_\mu(\cdot;\frac{2}{\beta}). \tag{2.19}$$

On the other hand, Goulden and Jackson had introduced in [GJ96] the *hypermap series*:

$$\begin{aligned}\Psi(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z};t,1+b) &= (1+b)t\partial_t\ln\sum_{\mu}\frac{J_\mu(\boldsymbol{x};1+b)J_\mu(\boldsymbol{y};1+b)J_\mu(\boldsymbol{z};1+b)}{\langle J_\mu,J_\mu\rangle_\alpha}t^{|\mu|}\\ &= \sum_{n\geq 1}t^n\sum_{\lambda,\mu,\nu\vdash n}h_{\mu\nu}^{\lambda}(b)p_\lambda(\boldsymbol{x})p_\mu(\boldsymbol{y})p_\nu(\boldsymbol{z}).\end{aligned}$$

The coefficient in the power-sum basis, the $(h_{\mu\nu}^{\lambda})$ can be interpreted as number of orientable hypermaps when $b=0$ and possible non-orientable hypermaps when $b=1$, with hyperedge, vertex, and face profile given by λ,μ,ν respectively. This led Goulden and Jackson to conjecture that for all b, the coefficients of the Hypermap series in the power-sum basis count maps, weighted by a factor depending of their orientability.

Conjecture 2.5.8 (Hypermap-Jack conjecture [GJ96])**.**

$$h_{\mu\nu}^{\lambda}(b) = \sum_{\mathfrak{m}} b^{\vartheta(\mathfrak{m})},$$

where the sum is on possible non-orientable hypermaps $\mathfrak{m}$ *with hyperedge, vertex and face profile* λ,μ,ν *respectively, and* $\vartheta(\mathfrak{m})$ *is a measure of the non-orientability of the map* $\mathfrak{m}$*.*

If we restrict the terms of the series to the case where the hyperedge profile is (2^n) – that is, the case of maps – we obtain the related *map series*:

$$\begin{aligned}M(\boldsymbol{y},\boldsymbol{z};t,1+b) &= (1+b)t\partial_t\ln\sum_{\substack{\mu\\ |\mu|\text{ even}}}\frac{J_\mu(\boldsymbol{y};1+b)J_\mu(\boldsymbol{z};1+b)}{\langle J_\mu,J_\mu\rangle_\alpha}t^{|\mu|}[p_2^{|\mu|/2}]J_\mu(\cdot;1+b)\\ &= \sum_{n\geq 1}t^{2n}\sum_{\mu,\nu\vdash 2n}h_{\mu\nu}^{(2^n)}(b)p_\mu(\boldsymbol{y})p_\nu(\boldsymbol{z}).\end{aligned}$$

Using the result of Goulden, Harer, and Jackson, we can express the map series in terms of an expectation under the measure of the β-ensemble:

$$M(\boldsymbol{y}, 1_N; \sqrt{\frac{z}{N}}, \frac{2}{\beta}) = \frac{2}{\beta}\sqrt{z}\partial_t|_{t=\sqrt{z}}\nu_{\beta,0}^N\left[\mathrm{e}^{\frac{\beta}{2}\sum_{j\geq 1}\frac{t^j}{j}p_j(\boldsymbol{y})p_j(\boldsymbol{\lambda})}\right].$$

In particular, we get that the cumulants of the β-ensemble coincide with sums of coefficients $h_{\mu\nu}^{(2^n)}$: for all integer $n \geq 1$ and $\mu \vdash 2n$,

$$\sum_{\nu\vdash 2n} h_{\mu\nu}^{(2^n)}(\frac{2}{\beta}-1)N^{l(\nu)-n} = \left(\frac{\beta}{2}\right)^{l(\mu)}\kappa_{l(\mu)}\left(\bigotimes_{j=1}^{l(\mu)} p_{\mu_j}(\boldsymbol{\lambda})\right),$$

where the cumulant is with respect to the measure $\nu_{\beta,0}^N$. This computation gives one answer to the question of computing the cumulants of the β-ensemble. These sums of coefficients were computed by LaCroix [LaC09]. In particular, he showed the following result.

Theorem 2.5.9 (Marginal b-conjecture [LaC09, Corollary 4.17])**.** *Let $f, n \in \mathbf{N}^*$ and $\mu \vdash 2n$. The sum*

$$\sum_{\substack{\nu\vdash 2n\\ l(\nu)=f}} h_{\mu\nu}^{(2^n)}(b)$$

is a polynomial in b with integer coefficients. Given $l \in \mathbf{N}$, the coefficient of b^l is the number of rooted maps with f faces, vertex profile μ, and l twisted edges when decomposed by an iterative root deletion algorithm (see [LaC09, Section 4.1]).

The approach of LaCroix was based on inductions equations and tools from algebraic combinatorics. We may wonder if there could be another approach, based purely on random matrix theory techniques as for the cases $\beta \in \{1, 2, 4\}$. When $\beta \notin \{1, 2, 4\}$, it is not possible to use Wick calculus: a self-adjoint matrix model with independent Gaussian entries is not available. However, the β-ensemble coincide with the distribution of the eigenvalues of a tridiagonal random matrix defined by Dumitriu and Edelman. The computation of the cumulants of the β-ensemble using this model will be carried out in Chapter 4.

2.6 Computation of moments: induction relations

In Section 2.5, we related some averages of observables to numbers of (weighed) maps. A way to provide closed expressions for these numbers is to use induction relations. We discuss first the Harer-Zagier recurrence, and then the powerful method of Dyson-Schwinger equations. The Dyson-Schwinger equations are used in particular to show that some formal expansion of matrix models are in fact exact.

2.6.1 The Harer-Zagier recurrence

The computation of the moments in the case $\beta = 2$ was carried out by Harer and Zagier [HZ86] using an induction formula. They were not interested in theses moments per se, their aim was the computation of the orbifold Euler characteristic $\chi(\mathcal{M}_g^1)$ of $\mathcal{M}_g^1$, the moduli space of curves of genus $g \geq 1$ with a base point. Their proof relied first on a reduction of the computation of $\chi(\mathcal{M}_g^1)$ to the enumeration of a class of one-face maps (maps without vertices of degree one or two), and then to a computation of this number of maps using expectations under the GUE. The latter computation was carried out using the induction equation given in the following Theorem.

Theorem 2.6.1 (Harer-Zagier recursion [HZ86, Theorem 3])**.** *Let $k \geq 0, N \geq 1$ be two integers. We have*

$$\left\langle p_{(2k)}(\lambda_1, \ldots, \lambda_N)\right\rangle_{2,0}^N = N^{-k}(2k-1)!!c(k, N),$$

where $c(n,N)$ is defined by $c(0,N) = N, c(k,0) = 0$ and the recursion

$$c(k,N) = c(k,N-1) + c(k-1,N) + c(k-1,N-1). \tag{2.20}$$

The ordinary generating function of the $(c(k,N))_{k\geq 0}$ is

$$1 + 2\sum_{k\geq 0} c(k,N)x^{k+1} = \left(\frac{1+x}{1-x}\right)^N.$$

Together with Proposition 2.5.2, Theorem 2.6.1 allows us to compute the number of one-face maps of any genus, with any number of edges.

2.6.2 Dyson-Schwinger equations

We introduce the Dyson-Schwinger equations, which will be used in Chapters 3 and 5. These equations rely on the invariance by translation of the reference measure of the model, of the Lebesgue and Haar measures in our case. Equivalently, it corresponds to using the fact that integrating a derivative gives 0 when the functions we consider vanish at infinity.

Hermitian matrix case. Let us study the derivation of non-commutative polynomials with respect to a coefficient of a matrix. In the case of one Hermitian matrix X, for all $k \in \mathbf{N}$ and $i, j, i', j' \in [N]$ we have

$$\frac{\partial}{\partial X_{i',j'}}(X^k)_{i,j} = \sum_{p=1}^{k} (X^{p-1})_{i,i'}(X^{k-p})_{j',j}.$$

At this point, it is natural to see the product of coefficients in the sum above as the coefficient of the simple tensor $X^{p-1} \otimes X^{k-p}$. Let us fix our notation. We write the elementary tensor matrix in $\mathrm{M}_N(\mathbb{C})^{\otimes 2}$ for the indices i, j, i', j' as

$$E_{\substack{i,j\\ i',j'}} = E_{i,j} \otimes E_{i',j'},$$

and for a pair of matrices of size $N \times N$, A and B,

$$(A \otimes B)_{\substack{i,j\\ i',j'}} = A_{i,j}B_{i',j'},$$

so that

$$A \otimes B = \sum_{i,j,i',j'=1}^{N} A_{i,j}B_{i',j'}E_{\substack{i,j\\ i',j'}}.$$

Two simple tensors $A \otimes B$ and $A' \otimes B'$ may be multiplied:

$$(A \otimes B) \times (A' \otimes B') = (AA') \otimes (BB'),$$

and this definition extends by linearity. It is then natural to introduce the two following definitions.

Definition 2.6.2. *Let $\mathcal{A}$ be the algebra of non-commutative polynomials in $l+1$ indeterminates with complex coefficients $\mathcal{A} = \mathbb{C}\langle x, x_1, \dots, x_l\rangle$. We define the non-commutative derivative*

$$\partial_x \colon \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}$$

and the cyclic derivative

$$\mathcal{D}_x \colon \mathcal{A} \to \mathcal{A}$$

as follows. Given a monomial P, we set

$$\partial_x P = \sum_{P=QxR} Q \otimes R \text{ and } \mathcal{D}_x P = \sum_{P=QxR} RQ,$$

and we extend ∂_x and $\mathcal{D}_x$ by linearity to all polynomials in $\mathcal{A}$.

With this notation, we have

$$\frac{\partial}{\partial X_{i',j'}}(X^k)_{i,j} = \left(\partial_X X^k\right)_{\substack{i,i'\\ j',j}}.$$

This can be generalized as follows.

Lemma 2.6.3. *Let $P \in \mathbb{C}\langle x_1, \dots, x_l\rangle$, $X_1, \dots, X_l$ be $N \times N$ Hermitian matrices, $q \in [l]$, and $i, j, i', j' \in [N]$. We have*

$$\frac{\partial}{\partial (X_q)_{i',j'}}\left(P(X_1, \dots, X_l)\right)_{i,j} = \left(\partial_{x_q} P\right)_{\substack{i,i'\\ j',j}}(X_1, \dots, X_l),$$

and

$$\frac{\partial}{\partial (X_q)_{i',j'}} \operatorname{Tr} P(X_1, \dots, X_l) = \left(\mathcal{D}_{x_q} P\right)_{j',i'}(X_1, \dots, X_l).$$

With this non-commutative derivative, we can state the Dyson-Schwinger equations in the Hermitian case.

Theorem 2.6.4. *Fix $N, m \in \mathbf{N}^*, p \in \mathbf{N}$, and a family of p deterministic matrices $A_1, \dots, A_p \in \mathrm{M}_N(\mathbb{C})$. For all $l \geq 1$ and $Q_1, \dots, Q_l \in \mathbb{C}\langle x_1, \dots, x_m, a_1, \dots, a_m\rangle$ set*

$$\mathcal{W}^N_{V,l}(Q_1, \dots, Q_l) = \kappa_l(\operatorname{Tr} Q_1, \dots, \operatorname{Tr} Q_l),$$

where the cumulants are under the measure $\mu^N_{2,V}$ and the polynomials are evaluated in the random matrices $X_1, \dots, X_m, A_1, \dots, A_p$.

For all $l \geq 1$ and $k \in [m]$, we have

$$\begin{aligned}
N^{l-1} \sum_{I \sqcup J = [l-1]} &\mathcal{W}^N_{V,\#I+1}\left(\bigotimes_{i \in I} P_i \otimes \mathrm{Id}\right) \otimes \mathcal{W}^N_{V,\#J+1}\left(\bigotimes_{j \in J} P_j \otimes \mathrm{Id}\right)(\partial_k P_l) \\
&+ N^{l-1}\mathcal{W}^N_{V,l+1}\left(P_1 \otimes \cdots \otimes P_{l-1} \otimes \partial_k P_l\right) \\
&= N^l \mathcal{W}^N_{V,l}\left(P_1 \otimes \cdots \otimes P_{l-1} \otimes (\mathcal{D}_k V + X_k) P_l\right) \\
&+ N^{l-1} \sum_{j=1}^{l-1} \mathcal{W}^N_{V,l-1}\left(P_1 \otimes \cdots \otimes \check{P}_j \otimes \cdots \otimes P_{l-1} \otimes (\mathcal{D}_k P_j) P_l\right),
\end{aligned} \tag{2.21}$$

where $\check{\cdot}$ means that the term is omitted.

To illustrate how these equations may be used, we consider the first Dyson-Schwinger equation, obtained by taking $l = 1$ in (2.21), and we divide it by N^2:

$$\frac{1}{N}\mathcal{W}^N_{V,1} \otimes \frac{1}{N}\mathcal{W}^N_{V,1}(\partial_k P) + \frac{1}{N^2}\mathcal{W}^N_{V,2}(\partial_k P) = \frac{1}{N}\mathcal{W}_{V,1}\left((\mathcal{D}_k V + X_k)P\right).$$

In the large N limit, it turns out that the term $\frac{1}{N^2}\mathcal{W}^N_{V,2}(\partial_i P)$ is negligible. Let us assume furthermore that the matrices $(A_j)_{j \in [p]}$ converge in distribution, i.e. that there exists

$$\tau_{\boldsymbol{a}} \colon \mathbb{C}\langle a_1, \dots, a_p\rangle \to \mathbb{C}$$

such that for all $Q \in \mathbb{C}\langle a_1, \dots, a_p\rangle$:

$$\lim_{N \to \infty} \frac{1}{N}\mathcal{W}^N_{V,1}(Q) = \lim_{N \to \infty} \frac{1}{N}\operatorname{Tr}(Q)(A_1, \dots, A_p) = \tau_{\boldsymbol{a}}(Q).$$

We are led to consider the equation with “boundary condition” specified by $\tau_{\boldsymbol{a}}$:

$$\begin{cases} \tau \otimes \tau(\partial_k P) = \tau\left((\mathcal{D}_k V + X_k)P\right), \text{ for all } P \in \mathbb{C}\langle x_1, \dots, x_m, a_1, \dots, a_p\rangle, \\ \tau|_{\mathbb{C}\langle x_1, \dots, x_m, a_1, \dots, a_p\rangle} = \tau_{\boldsymbol{a}}. \end{cases} \tag{2.22}$$

where τ is a linear map $\mathbb{C}\langle x_1, \dots, x_m, a_1, \dots, a_p\rangle \to \mathbb{C}$.

This equation has a unique solution if V is in some sense, small enough. As an example consider the case $V = 0$. If we denote by $\deg_k P$ the the degrees of P in x_k (recall Definition 2.2.4), we see that the left-hand side in the first line of (2.22) is in terms of polynomials whose degree is $\deg_k P - 1$, while the right-hand side is expressed in terms of a polynomial whose degree is $\deg_k P + 1$. By induction, the values of τ on monomials of positive degree is specified by the value of τ on polynomials of total degree 0. As it is, this argument fails as soon as $V \neq 0$, but we have the following Theorem.

Theorem 2.6.5 ([GM06, Theorem 2.1]). *Let $V \in \mathbb{C}\langle x_1, \dots, x_m, a_1, \dots, a_p\rangle$. It can be written as*

$$V = \sum_{i=1}^{d} z_i q_i$$

with $d \geq 1$ an integer, (z_i) complex numbers, and (q_i) monic monomials.

There exists $\epsilon > 0$ and $R > 0$ such that if

$$\max_{i \in [d]} |z_i| < \epsilon,$$

then the problem (2.22) has at most one solution τ satisfying for all $k \geq 1$

$$\max_{P} |\tau(P)| < R^k,$$

where the maximum is on monic monomials P with $\sum_i \deg_{x_i} = k$.

By a combinatorial argument generalizing the derivation of Tutte's equation of Section 2.4.1, we can show that generating series of (weighted) planar maps satisfy (2.22). The uniqueness of the solution of this problem provided by Theorem 2.6.5 implies that the first order asymptotics of the Hermitian one-matrix model we consider is given by a generating series of maps. This is the result obtained by Guionnet and Maurel-Segala in [GM06]. This result generalizes to the case of a nonzero (yet small) potential V the first order of the result obtain with Wick calculus in Proposition 2.5.1.

β-ensemble case. When $\beta \in \{1, 2, 4\}$, rather than deriving the Dyson-Schwinger equation at the level of the matrix entries, we may derive them at the level of the eigenvalues. In particular, this allow to obtain Dyson-Schwinger equations for the β-ensemble for any $\beta > 0$. Such an equation was obtained by Johansson [Joh98] in the study of the fluctuation of the eigenvalues of the β-ensemble. A convenient way to obtain it is to consider the n-points functions, which can be seen as ordinary generating functions of the cumulants:

$$W^N_{\beta,V,l}(z_1, \dots, z_l) = \kappa_l \left(\bigotimes_{i=1}^{l} \frac{1}{N} \sum_{j=1}^{N} \frac{1}{z_i - \lambda_j} \right) \text{ for } z_1, \dots, z_l \in \mathbb{C} \setminus \mathbb{R}.$$

Note that for $l = 1$, this is the expectation of the Stieltjes transform of the empirical measure of the particles of the β-ensemble. The Dyson-Schwinger equations are then obtained by making the change of variable

$$\lambda_i \mapsto \lambda_i - \frac{\epsilon}{z - \lambda_i}$$

for some small real parameter ϵ, differentiating with respect to ϵ, and then taking $\epsilon = 0$.

Theorem 2.6.6 (Dyson-Schwinger equation for the β-model)**.** *Let $l \in \mathbf{N}$ and $z, z_1, \ldots, z_l \in \mathbb{C} \setminus \mathbb{R}$. We have*

$$\begin{aligned}
&\sum_{I \sqcup J \subset [l]} W_{\#I+1}(z, \boldsymbol{z}|_I) W_{\#J+1}(z, \boldsymbol{z}|_J) \\
&- \kappa_{l+1} \left(\frac{1}{N} \sum_{j=1}^{N} \frac{V'(\lambda_i) + \lambda_i}{z_i - \lambda_j} \otimes \bigotimes_{i=1}^{l} \frac{1}{N} \sum_{j=1}^{N} \frac{1}{z_i - \lambda_j} \right) (V'(z) + z) \\
&+ \frac{2}{\beta} \sum_{i=1}^{l} \partial_{z_i} \left(\frac{W_l(z, z_1, \ldots, \check{z}_i, \ldots, z_l) - W_l(z_1, \ldots, z_l)}{z - z_i} \right) \\
&= (\frac{2}{\beta} - 1) \partial_z W_{l+1}(z, z_1, \ldots, z_l) - W_{l+2}(z, z, z_1, \ldots, z_l).
\end{aligned} \tag{2.23}$$

Unitary matrices case. Finally, the Dyson-Schwinger equations can be written for measure having a density with respect to the Haar measure on the unitary group. The idea is as follows: the Haar measure is invariant under translation so if U is Haar-distributed, we can make the change of variable

$$U \mapsto U \mathrm{e}^{i\epsilon M},$$

where M is an Hermitian matrix. After differentiating with respect to ϵ and setting $\epsilon = 0$ we obtain a family of equations. Consider a monomial $P \in \mathbb{C}\langle u, u^*, a_1, \ldots, a_p \rangle$, a Haar-distributed unitary matrix U, and a family of p deterministic matrices $A_1, \ldots, A_p$. We have

$$\begin{aligned}
0 &= \int_{\mathbb{U}_2(N)} \mathrm{d}U \frac{\partial}{\partial \epsilon}|_{\epsilon=0} \left(P(U\mathrm{e}^{i\epsilon M}, \mathrm{e}^{-i\epsilon M} U^*, A_1, \ldots, A_p) \right)_{ij} \\
&= \int_{\mathbb{U}_2(N)} \mathrm{d}U \sum_{P = QuR} (QUMR)_{ij} - \sum_{P = Qu^*R} (QMU^*R)_{ij}.
\end{aligned}$$

By considering matrices M which are basis elements of $\mathcal{H}_2(N)$, and doing appropriate linear combinations, we can assume that $M = (M_{pq})$ is such that $M_{pq} = \delta_{p,k}\delta_{q,l}$ for some $k, l \in [N]$. We then have

$$0 = \int_{\mathbb{U}_2(N)} \mathrm{d}U \sum_{P = QuR} (QU)_{ik} R_{lj} - \sum_{P = Qu^*R} Q_{ik} (U^*R)_{lj}. \tag{2.24}$$

This motivates the introduction of the following logarithmic version of the non-commutative derivative.

Definition 2.6.7. *The logarithmic non-commutative derivative is the linear map*

$$\partial^U : \mathbb{C}\langle u, u^*, a_1, \ldots, a_p \rangle \to \mathbb{C}\langle u, u^*, a_1, \ldots, a_p \rangle^{\otimes 2}$$

defined on a monomial P by

$$\partial^U P = \sum_{P = QuR} Qu \otimes R - \sum_{P = Qu^*R} Q \otimes u^*R,$$

and extended to $\mathbb{C}\langle u, u^*, a_1, \ldots, a_p \rangle$ by linearity.

With this notation, (2.24) becomes

$$\int_{\mathbb{U}_2(N)} \mathrm{tr}^{\otimes 2}(\partial^U P)(U, U^*, A_1, \ldots, A_p) = 0,$$

for all $P \in \mathbb{C}\langle u, u^*, a_1, \ldots, a_p \rangle$.

Proceeding similarly, we may define a family of Dyson-Schwinger equations, analogous to those of (2.21). We postpone the discussion of these equations to Chapter 3.

2.7 Geometry and integrable systems

2.7.1 A geometric interpretation of the Dyson-Schwinger equations

The Dyson-Schwinger equation for a one-matrix model (2.23) (in the case $l = 0$) can be rewritten in terms of the Stieltjes transform of the empirical measure. When taking the limit of a large matrix dimension N, the Dyson-Schwinger equation becomes

$$W_1(x)^2 - V'(x)W(x) + P(x) = 0, \tag{2.25}$$

where W_1 is the Stieltjes transform of the limit distribution of the eigenvalues, V is the potential of the matrix model, and P is a polynomial. Equation (2.25) can be interpreted as the equation of a hyperelliptic curve. We thus associate to the matrix model a complex curve, equipped with a distinguished 1-form constructed from W_1, the *spectral curve* of the matrix model. This idea was first used by Dijkgraaf and Vafa in the context of mirror symmetry [DV02]. It was subsequently used by Eynard [Eyn05] to re-express the hierarchy of Dyson-Schwinger equations in terms of computation of residue on the spectral curve. This point of view was enriched by works of Eynard and Orantin [EO07a; EO08]. Written in this framework, dubbed the *topological recursion*, the induction equations could be shown to describe many sequences of numbers of combinatorial or geometric nature: the topological recursion describes the enumeration of maps[CE06], intersection numbers[Eyn14] and Weil-Petersson volumes [Mir07; EO07b] on the moduli space of curves, (monotone) Hurwitz numbers [Bor+11; DDM14; Ale+18; BG18]...

A main idea of topological recursion for matrix models is to use the dictionary between the observables of the random matrix model and geometric quantities on the spectral curve, see for instance [BE12, Section 9], to turn problems of random matrix theory into residue computations. In Chapter 5 (based on [BB24]), we cross the bridge in the other direction: identities of random matrix theory are transported to the world of algebraic geometry, to become formulae on hyperelliptic curves.

2.7.2 Riemann surfaces and theta functions

We recall a few notions of geometry, needed to introduce the theta functions – the main character of our formulae – and Fay's formula, to which our formulae resemble.

Cycles and fundamental domain

Consider a compact Riemann surface without boundary Σ. Assume that Σ is of genus g at least 1. It is at times convenient to work on the universal covering $\hat{\Sigma}$ of Σ. We choose a fundamental polygon in $\hat{\Sigma}$, a choice of fundamental domain for the deck transformations. A way to construct this polygon is to choose a symplectic basis of the homology group $H_1(\Sigma; \mathbf{Z})$, i.e. a basis $\mathcal{A}_1, \dots, \mathcal{A}_g, \mathcal{B}_1, \dots, \mathcal{B}_g$ such that their intersection numbers are

$$\mathcal{A}_i \cap \mathcal{A}_j = 0, \mathcal{B}_i \cap \mathcal{B}_j = 0, \mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j},$$

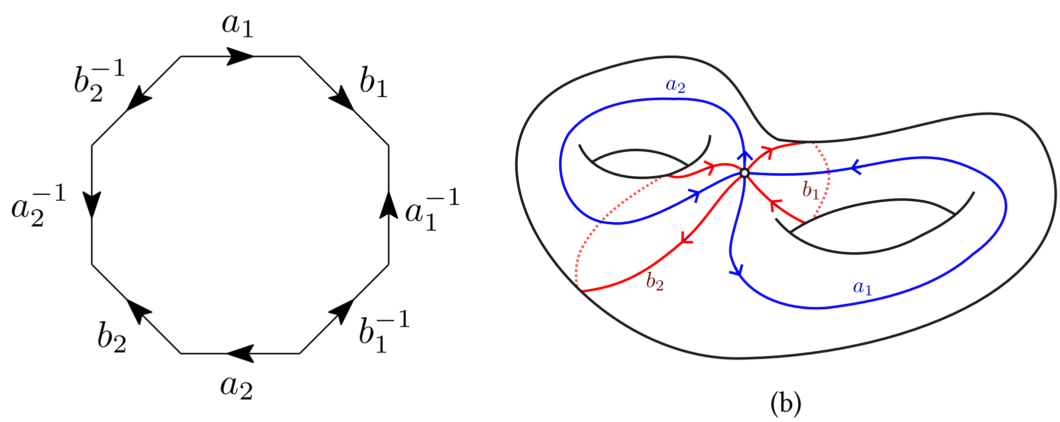
for $i, j \in [g]$. We can then choose a point $P_0 \in \Sigma$ and curves $a_1, \dots, a_g, b_1, \dots, b_g$ representing these classes such that all these curves intersect only at P_0. Cutting the surface Σ along the curves $(a_i, b_i)_{i\in[g]}$ gives a simply connected domain $\mathring{\Sigma}$, see Figure 2.6. Note that this polygon is by no mean unique.

By duality, the classes $(\mathcal{A}_i)_{i\in[g]}$ define a family of 1-forms $(\mathrm{d}u_i)_{i\in[g]}$. These forms are characterized by the normalization property on the $\mathcal{A}$-cycles

$$\oint_{\mathcal{A}_i} \mathrm{d}u_j = \delta_{i,j} \text{ for } i, j \in [g].$$

The integrals on the $\mathcal{B}$-cycles define the period matrix $\boldsymbol{\tau} = (\tau_{ij})_{i,j\in[g]} \in \mathrm{M}_g(\mathbb{C})$:

$$\tau_{ij} := \oint_{\mathcal{B}_i} \mathrm{d}u_j.$$



(a) (b)

Figure 2.6: (a) A fundamental polygon (b) The corresponding Riemann surface, of genus 2, with a choice of paths $a_i, b_i, i = 1, 2$.

An important property of the period matrix is that $\operatorname{Im} \boldsymbol{\tau}$ is a positive definite matrix. It is not uniquely defined as it depends on a choice of $\mathcal{A}$ and $\mathcal{B}$-cycles. However, any two choices are related by the action of a symplectic matrix with integer coefficients.

The period matrix allows us to define the Jacobi variety of Σ:

$$J(\Sigma) := \mathbb{C}^{2g} / \left(\mathbf{Z}^g \oplus \boldsymbol{\tau} \mathbf{Z}^g\right).$$

It is a complex torus determined by the curve Σ: two choices of period matrix for Σ yield the same Jacobi variety. Having fixed the point P_0, we can define the Abel map $\boldsymbol{u} = (u_i)_{i \in [g]} : \Sigma^\circ \to J(\Sigma)$: for all $z \in \Sigma^\circ$

$$u_i(z) = \int_{P_0}^{z} \mathrm{d}u_i,$$

where the integration is over a path contained in $\mathring{\Sigma}$. This mapping can be shown to be injective. We can naturally extend this map to a holomorphic map on the universal cover $\hat{\Sigma}$ of Σ. For our purpose, its interest is that it relate the Riemann surface Σ to its Jacobi variety, a much simpler variety since it is a complex torus (albeit of greater dimension $2g$).

Theta function

The theta function can be defined without any reference to geometry, as a Fourier series.

Definition 2.7.1. *Let $g \geq 1$ be an integer, and $\boldsymbol{\tau} \in \mathrm{M}_g(\mathbb{C})$ be such that $\operatorname{Im} \boldsymbol{\tau}$ is positive definite. The Riemann theta function is defined by*

$$\theta(\boldsymbol{z} \mid \boldsymbol{\tau}) = \sum_{\boldsymbol{m} \in \mathbf{Z}^g} \exp(i\pi \boldsymbol{m} \cdot \boldsymbol{\tau} \cdot \boldsymbol{m} + 2i\pi \boldsymbol{m} \cdot \boldsymbol{z}) \text{ for } \boldsymbol{z} \in \mathbb{C}^g.$$

Consider the particular case $g = 1$. As a function of z, the theta function can be considered as an elliptically deformed version of the trigonometric functions, see [KZ15] for a review of some of its properties.

Its key property is that it is a quasiperiodic function: for $\boldsymbol{n}, \boldsymbol{n'} \in \mathbf{Z}^g$ and $\boldsymbol{z} \in \mathbb{C}^g$,

$$\theta\left(\boldsymbol{z} + \boldsymbol{n} + \boldsymbol{\tau} \cdot \boldsymbol{n'} \mid \boldsymbol{\tau}\right) = \mathrm{e}^{-i\pi(\boldsymbol{\tau} \cdot \boldsymbol{n'} + 2\boldsymbol{z})} \theta\left(\boldsymbol{z} \mid \boldsymbol{\tau}\right).$$

We can thus consider it as "almost" a holomorphic function on the Jacobian variety $J(\Sigma)$. A geometric way to define the theta function is as (up to homothety) the section of a line bundle on $J(\Sigma)$.

On compact Riemann surfaces, non-constant holomorphic functions do not exist because of the maximum principle. This entails a lot of rigidity regarding which meromorphic function exist: a meromorphic function is determined by the position of its zeros and poles. Furthermore, it is not obvious that a meromorphic function with a given set of zeroes and poles do exist. A reason why the theta function is important is that it is a building block to construct meromorphic functions on compact Riemann surfaces. In particular, given $\boldsymbol{c} \in \mathbb{C}^g$, the function

$$\begin{cases} \mathring{\Sigma} & \to \mathbb{C} \\ P & \mapsto \theta\left(\boldsymbol{c} + \boldsymbol{u}(P) \mid \boldsymbol{\tau}\right) \end{cases}$$

is a holomorphic function of $\mathring{\Sigma}$ that could be extended to a holomorphic function on Σ if it were not for the phases by which it differs on the boundary of the fundamental polygon. A way to remedy to this problem is to construct a bispinor form using the theta function, the prime form E, which is a generalization of the function $(x, y) \mapsto x - y$ that can be defined on $\mathbb{C}^2$. We do not define precisely the prime form yet. It is a bispinor form $(x, y) \in \Sigma \mapsto E(x, y)$ whose only zero is at $x = y$. Ratios of prime forms can then be used to define meromorphic functions on Σ.

Hyperelliptic curves

We mentioned that Dyson-Schwinger equation (2.25) can be seen as the equation of an hyperelliptic curves. Hyperelliptic curves are Riemann surfaces that can be defined by an equation of the form

$$w^2 = Q(z)$$

with Q a polynomial, or equivalently, which are branched double coverings of the sphere. They can be described explicitly using two sheets which are copies of the Riemann sphere $\mathbb{CP}^1$, and a set of curves, the "cuts", by which the two spheres are connected. By crossing a cut, we go from one sheet to the other. While the cuts are not intrinsically defined, their endpoints are. They are exactly the roots of Q (when Q is a polynomial of even degree with only simple roots), or equivalently the Weierstrass points of the curve. The analysis is then greatly simplified: most computations can be written in terms of one of the two sheets only.

Finally, when considering only hyperelliptic curves rather than all complex curves, the deformations of curves are easier to describe. It suffices – for our purpose at least – to consider local deformation of the roots of the polynomial Q describing a hyperelliptic curve Σ: let $\alpha_1, \dots, \alpha_d$ be holomorphic functions from a neighborhood of 0 in $\mathbb{C}$ to $\mathbb{C}$, with $\alpha_1(0), \dots, \alpha_d$ the roots of Q. Let $Q_x(z) = \prod_{i=1}^{d}(z - \alpha_i(x))$ be a deformation of $Q = Q_0$. The polynomial Q_x defines a hyperelliptic curve Σ_x through:

$$w^2 = Q_x(z).$$

If Σ is equipped with a family of $\mathcal{A}$ and $\mathcal{B}$. cycles, for small enough deformation the deformed curve Σ_x is naturally equipped with a family of $\mathcal{A}$ and $\mathcal{B}$ cycles. The integrals on the $\mathcal{A}$ and $\mathcal{B}$ cycles are then holomorphic in the complex deformation parameter x. It follows in particular that the period matrix is a holomorphic function of the deformation parameter x. See [CMP17] for more.

2.7.3 Fay's formula and some applications

In [Fay73], Fay gave the following formula, in terms of the theta function θ, the Abel map $\boldsymbol{u}$, and the prime form E.

Theorem 2.7.2 (Fay's formula [Fay73, Corollary 2.19] or [Mum07, IIIb. §2])**.** *Let $\boldsymbol{e} \in \mathbb{C}^g$, and $n \geq 1$. For all $x_1, \dots, x_n, y_1, \dots, y_n \in \hat{\Sigma}$,*

$$\frac{\prod_{i<j} E(x_i, x_j) E(y_i, y_j)}{\prod_{i,j=1}^{n} E(x_i, y_j)} \theta\left(\sum_{i=1}^{n} (\boldsymbol{u}(x_i) - \boldsymbol{u}(y_i)) - \boldsymbol{e}\right) \theta\left(\boldsymbol{e}\right)^{n-1} = \det\left(\frac{\theta(\boldsymbol{u}(x_i) - \boldsymbol{u}(y_j) - \boldsymbol{e})}{E(x_i, y_j)}\right),$$

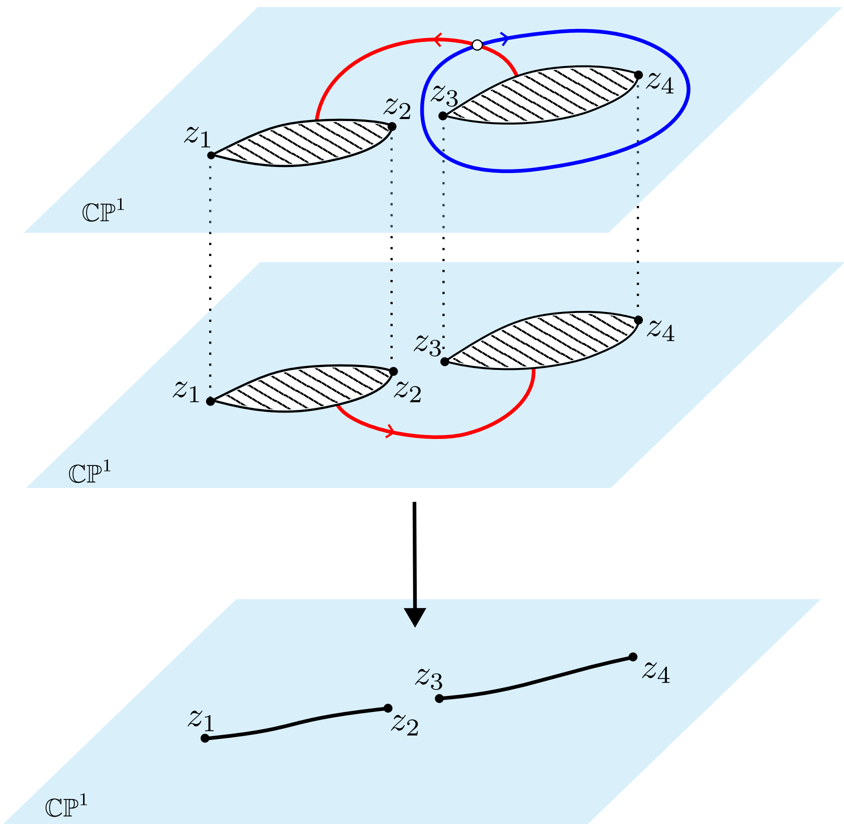


Figure 2.7: A hyperelliptic curve of genus 1 as a double covering, ramified at the Weierstrass points $z_1, \ldots, z_4$. Note that when arriving from one side of one of the cut on the upper sheet, we arrive in the lower sheet from the opposite side of the cut.

where θ is the theta function where $\boldsymbol{\tau}$ is a period matrix of Σ.

In fact, the general determinantal formula that we reproduced above can be deduced from the case $n = 2$:

$$\begin{aligned}\frac{E(x_1,x_2)E(y_1,y_2)}{E(x_1,y_1)E(x_1,y_2)E(x_2,y_1)E(x_2,y_2)}&\theta\left(\boldsymbol{u}(x_1)+\boldsymbol{u}(x_2)-\boldsymbol{u}(y_1)-\boldsymbol{u}(y_2)-\boldsymbol{e}\right)\theta\left(\boldsymbol{e}\right)\\ &=\frac{\theta(\boldsymbol{u}(x_1)-\boldsymbol{u}(y_1)-\boldsymbol{e})}{E(x_1,y_1)}\frac{\theta(\boldsymbol{u}(x_2)-\boldsymbol{u}(y_2)-\boldsymbol{e})}{E(x_2,y_2)}\\ &-\frac{\theta(\boldsymbol{u}(x_1)-\boldsymbol{u}(y_2)-\boldsymbol{e})}{E(x_1,y_2)}\frac{\theta(\boldsymbol{u}(x_2)-\boldsymbol{u}(y_1)-\boldsymbol{e})}{E(x_2,y_1)}.\end{aligned}\tag{2.26}$$

We now consider three applications of this formula, in geometry and the theory of integrable systems.

Fay's formula as a trisecant identity

The Jacobi variety $J(\Sigma)$ being a complex torus, we can define on it the mapping $\sigma\colon z \mapsto -z$. The Kummer variety of $J(\Sigma)$ is then the variety with singularities $K(\Sigma) = J(\Sigma)/\sigma$. Its singularities are at the points corresponding to the lattice $\frac{1}{2}\mathbf{Z}^g + \frac{1}{2}\tau\mathbf{Z}^g$. As a consequence of Lefschetz' theorem, the theta function allows us to define an embedding of the Kummer variety $K(\Sigma)$ in a projective space $\mathbb{CP}^{2^g-1}$.

Definition 2.7.3 (Secant of a Jacobian variety)**.** *Let $n \geq 3$ be an integer. A n-secant of the Jacobian variety $J(\Sigma)$ is a projective plane $V \simeq \mathbb{CP}^{n-2}$ of dimension $n-2$ which intersect the Kummer variety $K(\Sigma)$ at n distinct points.*

It was noticed by Mumford [Mum07] that Fay's formula (for $n = 2$) can be interpreted as the equation of a trisecant. We now concisely explain why, for a detailed account of this see the review of Taimanov [Tai97]. We remarked that the theta function is a building block of an embedding of $K(\Sigma)$ in $\mathbb{CP}^{2^g-1}$. The equation of a trisecant – if it exists – can thus be given in terms of theta functions. Given $z_1, z_2, z_3 \in J(\Sigma)$, there is a trisecant going through the image in projective space of the three points z_1, z_2, z_3 if and only if there exists nonzero coefficients c_1, c_2, c_3 such that

$$c_1\theta(z + z_1)\theta(z - z_1) + c_2\theta(z + z_2)\theta(z - z_2) + c_3\theta(z + z_3)\theta(z - z_3) = 0 \text{ for all } z \in \mathbb{C}^g.$$

For appropriate choices of z_1, z_2, z_3 (in terms of $\boldsymbol{e}$ and $\boldsymbol{u}(x_i), \boldsymbol{u}(y_i), i = 1, 2$), Fay's formula (2.26) with $n = 2$ can be expressed in this way, thus giving the equation of a trisecant.

Integrable systems

Integrable systems are families of partial differential equations whose flows commute. Beyond their intrinsic interest, these hierarchies of equations appear both in combinatorics and in random matrix theory. We give an informal description of some part of the theory that is relevant to this Thesis. For a thorough description of the theory see [BBT10], or the gentler introduction [MJD02]. Many integrable systems can be written in the following setup. Let L be a differential operator, or rather a *pseudodifferential operator*, which is a formal extension of differential operator allowing for negative powers of differentials. We may be interested in the eigenvalues and eigenfunctions of this operator, and hence study the equation

$$L\psi = k\psi.$$

The operator L and the eigenfunction ψ may have additional symmetries, that could be encoded into additional differential operators $(A_n)_{n\geq 1}$. The eigenfunction ψ thus depend on an infinite number of parameters, the "times" $\boldsymbol{t} = (t_1, t_2, \ldots)$, such that

$$\frac{\partial \psi}{\partial t_n} = A_n \psi.$$

The differential operators $(A_n)_{n\geq 1}$ are well-chosen, so that the flows of these equations are compatible. We do not make this precise and refer the reader to the references above. The important point for us is that integrable systems such as the Korteweg-de Vries (KdV), Kadomstev-Petviashvili (KP), or Toda hierarchies can be written in the form

$$\begin{cases} L\Psi = k\Psi \\ \frac{\partial \Psi}{\partial t_j} = A_j \Psi, \text{ for } j \geq 1. \end{cases} \tag{2.27}$$

In this form, the eigenfunction Ψ is a function of the so-called spectral parameter k and of an infinite number of times $\boldsymbol{t} = (t_1, t_2, \ldots)$. A solution of (2.27) is called a *Baker-Akhiezer* function. This function can be expressed in terms of another function of the times $\boldsymbol{t}$, the tau function. The integrable system equations (2.27) can the be recast in term of an equation on the tau function, the Hirota bilinear equation.

Theta functions appear in a crucial way in the algebro-geometric approch to integrable systems, due in particular to Krichever [Kri77]. The construction starts with geometric data, and produces an integrable system together with its Baker-Akhiezer function. More precisely, to a compact Riemann surface Σ and a choice of punctures $P_1, \ldots, P_n$ in Σ, we can construct uniquely a function Ψ: it is the unique meromorphic function on $\Sigma \setminus \{P_1, \ldots, P_n\}$ having a particular behavior around the punctures. The function Ψ can be expressed in terms of theta functions. A family of operators with appropriate commutation relation can then be constructed, such that their Baker-Akhiezer function is Ψ. This procedure allows the construction of quasiperiodic solutions to many integrable systems.

On the other hand, the quasiperiodic solutions constructed by the method of Krichever could be constructed independently by Mumford. Indeed, he noticed that degenerations of Fay's formula (i.e. corollaries obtained by taking limits in the points x_i, y_i) lead to equations of some integrable systems.

In this way, he constructed solutions to the KdV, KP, and Sine-Gordon equations [Mum78]. In fact, in many situations such as for the KdV, KP or Toda hierarchies, Hirota equation can be rephrased as Fay's formula, see for example the recent review [EO24].

The Schottky problem

The Fay identity appears in the Schottky problem, an old problem in geometry:

Question 2.7.4 (Schotty problem). Among complex matrices τ with positive definite imaginary part, which ones are period matrices of a Riemann surface?

The first results, in genus 4, were obtained by Schottky in 1888 [Sch88]. See the review by Grushevsky [Gru12] and reference therein for more on this problem. Beyond the relevance of this question in geometry, it appears in the theory of integrable systems under the guise of Novikov's conjecture which characterizes Riemann theta functions associated to Riemann surfaces (and hence period matrices) in terms of the KP hierarchy. This conjecture was proved by Shiota [Shi86], by refining the approach and results of Mulase [Mul84]. He showed that Fay's identity gives a characterization of period matrices, giving an answer to Question 2.7.4.

2.7.4 Integrable formulae for the β-ensemble

The β ensembles for $\beta \in \{1, 2, 4\}$ are also known for supplying solutions to some integrable systems. For $\beta = 2$, the partition function is a tau function of the Toda hierarchy [Ger+91]. For $\beta = 1$ or $\beta = 4$, the partition function is a tau function of the Pfaff lattice introduced by Adler, Horozov, and van Moerbeke [AHvM99], and studied by Adler, Shiota, and van Moerbeke [ASM02]. These links explain why observable of these matrix model enjoy many determinantal ($\beta = 2$) and pfaffian formulae ($\beta = 1, 4$).

At the root of this link is the description of the β-model in terms of orthogonal polynomials ($\beta = 2$) and skew-orthogonal polynomials ($\beta = 1, 4$). Orthogonal polynomials are a powerful way to study the β-ensemble at $\beta = 2$. Fix $N \geq 1$ and a confining polynomial potential V. The associated sequence of orthogonal polynomials $(p_{N,j})_{j\geq 0}$ is the sequence of monic polynomials, with $p_{N,j}$ of degree j for $j \in \mathbf{N}$, which are orthogonal with respect to the scalar product

$$\langle P, Q \rangle = \int \overline{P(x)} Q(x) \mathrm{e}^{-NV(x) - Nx^2/2} \mathrm{d}x,$$

that is

$$\langle p_{N,i}, p_{N,j} \rangle = \delta_{i,j} h_{N,i},$$

for some positive real numbers $(h_{N,i})_{i\geq 0}$. Gerasimov, Marchakov, Mironov, Morozov, and Orlov [Ger+91], showed that the recurrence equations satisfied by the orthogonal polynomials where equivalent to the equation of the Toda hierarchy together with the Virasoro constraints (equivalent to the the Dyson-Schwinger equations in this setting). A similar discussion involving the recurrence equation of skew-orthogonal polynomials allowed to show a corresponding result linking the Pfaff lattice and the cases $\beta = 1$ and $\beta = 4$ [ASM02].

The particular observable we are interested in are averages of ratio of characteristic polynomials of the random matrix, i.e. quantities of the form

$$\left\langle \frac{\prod_{i=1}^{n} \det(x_i - X)}{\prod_{j=1}^{m} \det(y_j - X)} \right\rangle_{\beta,V}^{N}, \tag{2.28}$$

for two integers $m, n \geq 0$ and $x_1, x_n, y_1, \ldots, y_m \in \mathbb{C} \setminus \mathbb{R}$. These observable encode much information about the distribution of the β-ensemble. For $m = n = 1$, such ratio are related to the Stieltjes transform

$$\frac{\mathrm{d}}{\mathrm{d}z} \left\langle \frac{\det(z - X)}{\det(z' - X)} \right\rangle_{\beta,V}^{N} |_{z=z'} = \left\langle \mathrm{Tr}\, \frac{1}{z - X} \right\rangle_{\beta,V}^{N} = NW_1(z).$$

For $\beta = 2$, we recover the orthogonal polynomials in the case $n = 1, m = 0$.

Proposition 2.7.5 (Heine's formula, see for instance [EKR18, Equation (5.2.9)]). *Let V be a confining potential. The k-th orthogonal polynomial associated to the potential V is*

$$z \mapsto \langle \det(z - X) \rangle_{2,V}^{k} .$$

Formulae relating general ratio of the form (2.28) to a determinant ($\beta = 2$) or or a Pfaffian of simpler ratio involving only to characteristic polynomials where obtained by Borodin and Strahov [BS06]. They we be introduce in Chapter 5.

2.8 Asymptotics of matrix models

We now discuss asymptotics of matrix models, starting from the topological expansion for one and multi-matrix models in the case $\beta = 2$. We then discuss the asymptotic expansion of models of unitary matrices, to introduce the topological expansion derived in Chapter 3. We then discuss the asymptotics of the β-ensemble in the multicut regime – the case where the limiting eigenvalue distribution (when $N \to \infty$), the equilibrium measure, has a disconnected support.

2.8.1 Topological expansions

What is a topological expansion?

Consider the multi-matrix models introduced in Section 2.3.2, with $m \geq 1$ random matrices and a potential V. Recall that we defined the partition function by

$$Z_{2,V}^{N} = \int_{\mathcal{H}_N^m} \exp\left(-N \operatorname{Tr} V(X_1, \ldots, X_m) - \frac{N}{2} \sum_{i=1}^{m} \operatorname{Tr} X_i^2 \right) \mathrm{d}X_1 \cdots \mathrm{d}X_m. \tag{2.29}$$

A simple example which we encountered before is the quartic model, i.e. the choice $m = 1$, $V(X) = \lambda X^4$ for $\lambda \geq 0$:

$$Z_{2,gX^4}^{N} = \int_{\mathcal{H}_N} \mathrm{e}^{-N\lambda \operatorname{Tr} X^4 - \frac{N}{2} \operatorname{Tr} X^2} \mathrm{d}X. \tag{2.30}$$

We explained in Section 2.5.1 how the free energy

$$F_{2,V}^{N} = \frac{1}{N^2} \ln \frac{Z_V^N}{Z_0^N} \tag{2.31}$$

can be expressed at the level of formal power series as an expansion in powers of $1/N$:

$$F_{2,V}^{N} = \frac{1}{N^2} \ln \frac{Z_V^N}{Z_0^N} = \mathcal{M}_0(V) + N^{-2}\mathcal{M}_1(V) + N^{-4}\mathcal{M}_2(V) + \cdots . \tag{2.32}$$

In the above expression, the functions $\mathcal{M}_g$ are formal series in the coefficients of V. They are generating function of maps. For instance, in the example (2.30), $\mathcal{M}_g$ is a generating function in the parameter λ of numbers of quadrangulations (maps with vertices of degree 4) of genus g. Because of the importance of the genus of the maps, such an expansion is called a formal *topological* expansion.

Let us interest ourselves in the analytic large N expansion of the free energy. We would like an expansion such as (2.32) to hold at the level of convergent series rather than formal series. There are thus two related questions:

Question 2.8.1. Is there an asymptotic expansion of the form

$$F_{2,V}^{N} = C_0(V) + N^{-2}C_1(V) + N^{-4}C_2(V) + \cdots + N^{-2g}C_g(V) + \mathcal{O}\big(N^{-2g-2}\big), \tag{2.33}$$

for some coefficients C_i that depends on the potential V but not on N? Are the C_i analytic functions of the coefficients of V?

Question 2.8.2. Can the coefficients C_g be identified with the coefficients $\mathcal{M}_g$ of the formal expansion?

If the two questions can be answered positively, we have shown that $F^N_{2,V}$ admits an asymptotic topological expansion.

Heuristically, it is expected that such an expansion holds rigorously only when V has small coefficients, in some sense to be determined later. Once this expansion is established, the main question is to determine the properties of the coefficients $(\mathcal{M}_h)_{h\geq 0}$. For instance a question of interest in 2D gravity is: do these functions have singularities, and if so what are their critical exponents? See [DGZ95] for a discussion of this question.

In the situations we consider, we have a topological expansion not only for the free energy, but also for its derivatives. As we saw in Section 2.2.2, by differentiating the free energy (2.31) we obtain the cumulants of observables of the matrix model. We will also consider the expansion of the joint cumulants of variables $\operatorname{Tr} P_1, \ldots, \operatorname{Tr} P_l$ for $P_1, \ldots, P_l$ non-commutative polynomials under the measure μ^N_V. We denote these cumulants by

$$\mathcal{W}^N_{V,l}(P_1, \ldots, P_l) = \kappa_l(\operatorname{Tr} P_1, \ldots, \operatorname{Tr} P_l) \quad (\text{under } \mu^N_V). \tag{2.34}$$

The one-matrix model and orthogonal polynomials

A first rigorous derivation of a topological expansion for a family of one-matrix models (the case $m = 1$) was obtained by Ercolani and McLaughlin [EM03]. When the potential V is small enough (which implies in particular that the equilibrium measure of the model has a connected support): both Questions 2.8.1 and 2.8.2 can be answered positively. The expansion (2.33) holds, with coefficients $C_g(V)$ which are analytic functions in the coefficients of V in a neighborhood of 0. They are the generating series of maps $\mathcal{M}_g(V)$ given in Section 2.5.1.

The proof of Ercolani and McLaughlin is based on the following remark. If we consider the partition function associated to the measure of the eigenvalues defined in Theorem 2.3.4,

$$\mathcal{Z}^N_{2,V} = \int_{\mathbb{R}^N} |\Delta(\lambda_1, \ldots, \lambda_N)|^2 \, \mathrm{e}^{-N\sum_{i=1}^N V(\lambda_i) - \frac{N}{2}\sum_{i=1}^N \lambda_i^2} \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N,$$

we notice that the Vandermonde determinant can be expressed in terms of the family of orthogonal polynomials $(p_{N,j})_{j\geq 0, N\geq 1}$ introduced in Section 2.7.4. The Vandermonde determinant can be rewritten in terms of the orthogonal polynomials:

$$\begin{aligned}
\mathcal{Z}^N_V &= \int_{\mathbb{R}^N} \det\left(p_{i,N}(\lambda_j)\right)^2 \mathrm{e}^{-N\sum_{i=1}^N V(\lambda_j) - \frac{N}{2}\lambda_j^2} \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N \\
&= \det_{1\leq i,j\leq N}\left(\langle p_{N,i}, p_{N,j}\rangle\right) \\
&= \prod_{i=1}^N h_{i,N}.
\end{aligned}$$

Thus, to understand the asymptotic expansion of $\hat{Z}^N_V$, it suffices to understand the asymptotic expansion of the numbers $h_{N,i}$. The key technique used in [EM03] is to remark that the orthogonal polynomials are characterized as the building blocks of a matrix-valued holomorphic function which is the unique solution to a Riemann-Hilbert problem. More precisely, the key result is the following consequence of a theorem due to Fokas, Its, and Kitaev [FIK91] (subsequently used in the series of papers [Dei+97; Dei+99b; Dei+99a]).

Theorem 2.8.3 (Particular case of [Dei+99a, Theorem 3.1])**.** *Let $N, n \geq 1$ be two integers. The matrix-valued function*

$$Y(z) = \begin{pmatrix} p_{n,N}(z) & \int_{\mathbb{R}} \frac{p_{n,N}(s) w_N(s)}{s-z} \frac{\mathrm{d}s}{2i\pi} \\ -2i\pi \frac{p_{n-1,N}(z)}{h_{n-1,N}} & -\int_{\mathbb{R}} \frac{p_{n-1,N}(z)}{h_{n-1,N}} \frac{w_N(s)}{s-z} \mathrm{d}s \end{pmatrix},$$

is the unique solution to the Riemann-Hilbert problem:

1. *$Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic;*

2.
$$Y_+(s) = Y_-(s) \begin{pmatrix} 1 & w_N(s) \\ 0 & 1 \end{pmatrix} \text{ for } s \in \mathbb{R}, \text{ with } Y_\pm(s) = \lim_{\epsilon \to 0^+} Y(s \pm i\epsilon);$$

3.
$$Y(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = \mathrm{Id} + \mathcal{O}\left(\frac{1}{|z|}\right) \text{ as } |z| \to \infty.$$

Applying several transformations to this Riemann-Hilbert problem and its solution, and comparing it with an appropriate approximation allow Ercolani and McLaughlin to compute the asymptotics of the orthogonal polynomials.

This Riemann-Hilbert technique has many applications in the study of orthogonal polynomials, and thus to random matrix theory. Recently, the Riemann-Hilbert method has been used to study the Ising model, mentioned in Section 1.2, by Duits, Hayford, and Lee [DHL25]. For the one-matrix model, the method relies on the fact the random matrix can be diagonalized, and the appearance of a Vandermonde determinant. In the sequel, we will consider general multi-matrix models, for which other methods are needed.

Multi-matrix models and the Dyson-Schwinger equations

At the formal level, multi-matrix models allow to describe richer families of maps. Indeed, the coefficients of the formal expansion – constructed analogously as in Section 2.5.1 – can be shown to be numbers of multicolored maps with a given vertex profile, genus, and coloration of the half-edges.

Specific cases of multi-matrix models may be investigated at an analytical level using biorthogonal polynomials [Meh81]. Large deviation techniques allow the study of the first order asymptotics of some important multi-matrix models [Gui04] (including, for instance, the Ising model discussed in Section 1.2). Here, we present a different and very general approach, based on the Dyson-Schwinger equations introduced in Section 2.6.2. In many cases, the Dyson-Schwinger equations allow to study all the orders of the $1/N$ asymptotic expansion. Thanks to their similarity with Tutte's equation, discussed in Section 2.4.1, the asymptotic expansion can be described in terms of (multicolored) maps. This approach is used in the work of Guionnet and Maurel-Segala to treat both the leading order of the asymptotics [GM06] of the cumulants (2.34), and the second order [GM07]. A description of all the orders of the expansion was subsequently obtained by Maurel-Segala in [Mau06].

To show the first-order asymptotics, the proof of Guionnet and Maurel-Segala is in two parts. First, the Dyson-Schwinger equation (2.22) is shown to have a unique solution when the potential V is small enough, and that this solution exists. Then, it is shown that the generating series of planar maps does satisfy the Dyson-Schwinger equations, which corresponds to Tutte equations. This shows that the cumulants coincide, when V is small, with a generating series of edge-colored planar maps.

To tackle the second-order asymptotics, a key idea is to rewrite the Dyson-Schwinger equation (2.21) in term of a linear operator Ξ acting on $\mathbb{C}\langle x_1, \ldots, x_m \rangle$ which is invertible on some appropriate completion of the space of non-commutative polynomials without constant term, when V is small enough. By inverting the operator (on a suitable space), the Dyson-Schwinger equations can be shown to have a unique solution and the second order of the asymptotics can be determined. Once this is done, it can be identified with a generating series of maps on the torus (genus $g = 1$).

The higher order terms of the expansion are treated similarly. The higher-order Dyson-Schwinger equations can be expressed in terms of the operator Ξ. When the inverse Ξ^{-1} exists, the higher order terms of the expansion can be obtained, and shown to coincide with generating series of maps of higher genera. Thus answering positively Questions 2.8.1 and 2.8.2.

Asymptotic expansion for integrals over the unitary group

The Dyson-Schwinger method generalizes to unitary matrix models. Collins, Guionnet and Maurel-Segala [CGM09] derived the Dyson-Schwinger equations on the orthogonal and unitary group. They used it to show that, again in the case of a small potential, the leading term of the asymptotics is a generating function of planar maps with “dotted edges” satisfying some conditions. The existence of the all-order asymptotic expansion was proved by Guionnet and Novak [GN15]. They used the same method as described in Section 2.8.1: they introduced a relevant operator, which can be inverted in some appropriate space, under the condition that V is small enough. The “gradient trick” on which relies the definition of this operator is explained in Section 3.9. When this operator can be inverted, the terms of the asymptotic expansion can be characterized as the unique solution of a family of Dyson-Schwinger equations.

The expansion of Guionnet and Novak answered Question 2.8.1 but not Question 2.8.2 – only the leading order was identified as a generating series of a family of maps. An insight into the combinatorics of unitary integrals is offered by the thorough study of the HCIZ integral by Goulden, Guay-Paquet, and Novak [GGN14; GGN11]. This integral is shown to be related to the so-called monotone double Hurwitz numbers defined in Section 2.4.2. The main question studied in Chapter 3 is to give a general description of the terms in the asymptotic expansion in terms of combinatorial maps. The maps we obtain, called maps of unitary type, can be seen as generalizations of monotone Hurwitz numbers.

2.8.2 Asymptotics of the β-model in the multicut regime

The asymptotics of the β-ensemble were described formally in the case $\beta = 2$ by Brézin, Itzykson, Parisi and Zuber, and later as an asymptotic expansion by Ercolani and McLaughlin, in the case where the equilibrium measure has a connected compact support[1]. For a general (confining) polynomial potential V, the equilibrium measure may have a disconnected support, made of a finite number of compact intervals. It has been noticed by Brézin and Deo [BD99] that in the case of a two-wells symmetric potential, the asymptotics of the correlators depended on the parity of the matrix dimension N, in contradiction with what could be expected in the one-cut case. This kind of oscillatory behavior had been observed numerically by [Jur91]. In the physics literature, an explanation to this puzzle was proposed by Bonnet, David, and Eynard [BDE00]: in the saddle point computations used to compute asymptotics of the free energy, it crucial to use the fact that the number of eigenvalues is an integers. For instance, for a two-cuts symmetric model, if N is even, the same number of eigenvalues may land in each of the two cuts. If N is odd, the two cuts contain a different number of eigenvalues. The two cases give different results. This “tunneling” behavior of the eigenvalues is mathematically modeled using theta functions whose argument depend on N. Slightly before Bonnet et al., this behavior was described in the mathematical literature: using Riemann-Hilbert methods, Deift, Kriecherbauer, McLaughlin, Venakides and Zhou [Dei+97; Dei+99b] computed the asymptotics of orthogonal polynomials associated to a general confining potential V. They found that the leading order of those asymptotics could be expressed in terms of theta functions. The link between orthogonal polynomials and partition functions described in Section 2.8.1 allows to recover in this way the asymptotics of partition functions of Hermitian one-matrix models. The form of the all-orders asymptotic expansion was proposed by Eynard [Eyn09]: the sub-leading order are expressed using derivatives of theta functions. The first sub-leading order was obtained rigorously by Shcherbina [Shc13; Shc14] for all $\beta > 0$. The full asymptotic expansion was made rigorous by Borot and Guionnet [BG24], for all $\beta > 0$.

An important consequence of the appearance of theta functions in the expansion, oscillating terms in N, is that there cannot be an asymptotic or topological expansion in the sense of Section 2.8.1 in the multi-cut case. In fact, different choices for the number of particles in each cut yield different solution to the Dyson-Schwinger equations. This implies that the method described in 2.8.1 fails as it relied on the uniqueness of the solution to the Dyson-Schwinger equations.

[1]We make a jump backward in time here: the early paper mentionned in this section date back to between 1997 and 2000, while the work of Ercolani and McLaughlin mentioned in Section 2.8.1 was published in 2003.

2.9 Localization and delocalization of eigenvectors

We finally turn to the problem of localization of eigenvectors, introduced in Section 1.7. We discuss first delocalization, and then review some techniques we use in Chapter 6.

2.9.1 A few remarks on delocalization

To make the link with the previous sections, we notice that unitarily-invariant models are not very interesting from the point of view of the localization phenomenon: if U^N is a $N \times N$ Haar-distributed unitary matrix, we can compute exactly the probability (see [HP14, Section 4.2])

$$\mathbb{P}\left(\left|U_{ij}^N\right|^2 > \frac{C \ln N}{N}\right) = \left(1 - \frac{C \ln N}{N}\right)^{N-1} \sim N^{-C} \text{ as } N \to \infty.$$

This ensures that the eigenvectors are delocalized with high probability, i.e. for all eigenvector $\boldsymbol{v}$ of a unitary invariant random matrix

$$\|\boldsymbol{v}\|_\infty = \mathcal{O}\left(\sqrt{\frac{\ln N}{N}}\right),$$

with high probability.

In general, showing that eigenvectors are delocalized may be a more involved process. A way to prove this is to obtain local laws. A local law is an approximation result for the resolvent $G(z) = (z - X)^{-1}$, with z allowed to have imaginary part close to 0 at a rate depending on N. More precisely, fix $\kappa > 0$ and $L > 0$, a closed set $S_\kappa \in \mathbb{R}$ that may depend on κ, and define the spectral domain

$$\mathcal{S}_{\kappa,N,L} = \left\{E + i\eta \in \mathbb{C} \colon E \in S_\kappa, \eta \in [N^{-1+\kappa}, L]\right\}.$$

A local law is then a result of the following form: there exists a matrix-valued function $m \colon \mathbb{C} \setminus \mathbb{R} \to \mathrm{M}_N(\mathbb{C})$ such that for all $x, y \in [N]$, for all $z \in \mathcal{S}_{\kappa,N,L}$

$$|G_{xy}(z) - m_{xy}(z)| = o(1). \tag{2.35}$$

Assuming that $\operatorname{Im} m_{xy}(z)$ is bounded by a constant $B > 0$, such a result implies delocalization. Denote by $w_1, \dots, w_N$ the eigenvectors of X and by $\lambda_1, \dots, \lambda_N$ the associated eigenvalues. For any i such that $\lambda_i \in S_\kappa$ and $z = \lambda_i + i\eta \in \mathcal{S}_{\kappa,N,L}$:

$$|\langle 1_x, w_i\rangle|^2 \le \eta \cdot \sum_{j\in[n]} \frac{\eta |\langle 1_x, w_j\rangle|^2}{\eta^2 + (\lambda_i - \lambda_j)^2} = \eta \operatorname{Im} G_{xx}(\lambda_i) \le \eta B.$$

Taking $\eta = N^{-1+\kappa}$ yields the delocalization result:

$$\|w_i\|_\infty^2 = \mathcal{O}\big(N^{-1+\kappa}\big).$$

2.9.2 Adjacency matrices of graphs

In the sequel, we consider adjacency matrices of random graph models. Adjacency matrices of random graphs display a very different structure from the invariant matrix ensembles discussed in Section 2.3. In general, their distribution is not invariant under the action of a particular group. In particular cases, they are indeed invariant under the action of the permutation group $\mathfrak{S}_N$ permuting the N vertices of the graph: in this case the model is *homogeneous*. It is the case of the Erdős-Rényi model whose definition was recalled in Section 1.7. Let us briefly discuss localization of eigenvectors for the adjacency matrix of the Erdős-Rényi model, as studied by Alt, Ducatez and Knowles [ADK23; ADK21b; ADK21a; ADK24]. Recall that they showed that there is localization-delocalization transition in the regime where

$$\sqrt{\ln N} \ll pN \le \mathcal{O}(\ln N).$$

The eigenvectors associated to large eigenvalues where shown to be localized, and those associated to small eigenvalues where shown to be delocalized. In particular, the delocalization result was proved by showing a local law such as (2.35). We now discuss some of the techniques used to prove localization. Adaptation of these techniques are used in Chapter 6: the random graph model we study can be seen as a generalization of the Erdős-Rényi model.

The heuristics used in [ADK21b; ADK21a] is that the biggest eigenvalues λ correspond to the vertices x of highest degree in the graph. Such vertices are rare, and surrounded with vertices whose degree concentrate around the average degree pN. Because the graph is sparse, when looking at the graph in a small ball (for the graph distance) around such a x, and up to removing a small number of edges, the graph is a (d_1, d_2)-regular tree: x is of degree d_1 and all the other vertices are either leafs or of degree d_2, with $d_2 \simeq pN$. The adjacency matrix of such a graph can be explicitly diagonalized. For instance, if we look in a ball of radius one, the eigenvectors associated to non-zero eigenvalues are of the form

$$v_{\pm}(x) = \frac{1}{\sqrt{2}} \left(1_x \pm \frac{1}{\sqrt{d_1}} 1_{S_1(x)} \right),$$

where $1_x = (\delta_{x,y})_{1 \le y \le N}$ is the vector with only its x-th component equal to 1 and $1_{S_1(x)} = \sum_y 1_y$ with the sum on the neighbors y of x in the graph. Up to an additional pruning of the graph – which only lightly perturb the adjacency matrix – these vectors $v_{\pm}(x)$, for x of high enough degree, form an orthonormal family.

This gives a family of candidate eigenvectors for the adjacency matrix. Each vector in the family is localized around a vertex of high degree. This family is then used to construct an approximation of the adjacency matrix and prove a localization result using a spectral gap argument, which we now describe.

Consider a matrix X, with eigenvalues (λ_i) and associated normalized eigenvectors (x_i). Let $G_X(z) = (z - X)^{-1}$ be its resolvent. Assume that we know that there is a spectral gap of size $\eta > 0$ around some value λ, i.e. that X has no eigenvalue in $[\lambda - \eta, \lambda + \eta]$. In that case, we can bound the operator norm of $G(\lambda)$:

$$\|G_X(\lambda)\| = \left\| \sum_i \frac{x_i x_i^*}{\lambda - \lambda_i} \right\| = \max_i \frac{1}{|\lambda - \lambda_i|} < \frac{1}{\eta}.$$

This can be used to show the localization as follows. Assume that we can show that:

- $v = v_+(x)$ is close to being an eigenvector of A with eigenvalue $\tilde{\lambda} = \sqrt{d_1}$, in the sense that

  $$Avv^* = \tilde{\lambda} vv^* + \epsilon,$$

  where ϵ is some small error;
- $\tilde{A} = (1 - vv^*)A(1 - vv^*)$ has a spectral gap around $\tilde{\lambda}$, of radius 2η.

For convenience define the two projectors

$$\Pi = vv^* \quad \text{and} \quad \overline{\Pi} = \mathrm{Id} - vv^*.$$

In particular $\tilde{A} = \overline{\Pi} A \overline{\Pi}$, and

$$\overline{\Pi} A \Pi = \tilde{\lambda} \overline{\Pi} \Pi + \overline{\Pi} \epsilon = \overline{\Pi} \epsilon, \tag{2.36}$$

the latter equation imply that we have an approximate block decomposition of A as $A \simeq \tilde{A} + \Pi A \Pi$. Then, if λ is an eigenvalue of A in $[\tilde{\lambda} - \eta, \tilde{\lambda} + \eta]$, with associated normalized eigenvector q, we have:

$$\overline{\Pi} q = G_{\tilde{A}}(\lambda) \left(\tilde{A} - \lambda \right) \overline{\Pi} q = G_{\tilde{A}}(\lambda) \overline{\Pi} \left(A \overline{\Pi} - \lambda \right) q = -G_{\tilde{A}}(\lambda) \overline{\Pi} \epsilon q,$$

where in the second equality we used the definition of $\tilde{A}$, in the third equality we used the definition of $\overline{\Pi}$, in the fourth equality we used the eigenvector-eigenvalue equation for A, and (2.36). Taking the 2-norm and using the spectral gap property, we obtain

$$1 - |\langle v, q \rangle|^2 \le \frac{1}{\eta^2} \|\epsilon\|^2,$$

this can be rewritten as

$$|\langle v, q \rangle|^2 \geq 1 - \frac{1}{\eta^2}\|\epsilon\|^2.$$

Hence, we obtain a localization result. A variation of this argument will be used in Chapter 6, to obtain Theorem 6.1.9.

We now discuss our first contribution, related to integrals over the unitary groups.

Chapter 3

Topological expansion of unitary integrals and maps

This Chapter is based on [Buc24].

3.1 Introduction

We saw in Section 2.8.1 that in the perturbative regime, the terms in the large N expansion of Hermitian matrix models are generating functions of (possible multicolored) maps. In this chapter, we establish a similar link between integrals of unitary matrices and the combinatorics of some maps. More precisely, we introduce new maps, the maps of unitary type (Definition 3.3.7), that describe the topological expansion. These maps allow us to relate the Weingarten calculus and the Dyson-Schwinger equation – two important ways to study unitary integrals. In a particular case, the maps of unitary type are related to Hurwitz numbers. In this way, we generalize part of the results obtained in [GGN11], that relate a particular integral, the HCIZ integral, to Hurwitz numbers.

The Haar unitary matrices share the same unitary invariance as matrices of the Ginibre ensemble. An expansion in terms of non-crossing permutations for expectation of traces of words of Ginibre matrices G_i has been obtained in [MN04]. This point of view in terms of non-crossing permutations is similar to the interpretation in terms of maps. For instance, the annuli considered in multi-annulus permutations correspond to the vertices in a map. A genus expansion in terms of maps has been obtained in [DP21]. They consider only some pairings, *admissible pairings*, which corresponds to an orientation of the edges of their maps. We recover such a feature in the maps of unitary type. In particular, the unitary invariance of the Ginibre ensemble implies that to have a non-zero expectation, the words in Ginibre matrices considered must be *balanced*, i.e. contain as many G_i as G_i^*. A similar condition appears for Haar unitary matrices.

We introduce some notation. Let $p \in \mathbf{N}^*$. For all $N \geq 1$, we fix p deterministic matrices $A_1^N, \ldots, A_p^N$ of size $N \times N$. The matrix U^N will be a unitary matrix of size $N \times N$, i.e. an element of the unitary group $\mathbb{U}(N)$, and $(U^N)^* = (U^N)^{-1}$ will be its conjugate transpose.

Let $\mathrm{d}U^N$ be the Haar measure on the unitary group $\mathbb{U}(N)$, and V be a non-commutative polynomial in several variables, that does not depend on N. The measure μ_V^N is given by

$$\mathrm{d}\mu_V^N(U^N) = \frac{1}{Z_V^N} \exp\left(N \operatorname{Tr} V\left(U^N, (U^N)^*, A_1^N, (A_1^N)^*, \ldots, A_p^N, (A_p^N)^*\right)\right) \mathrm{d}U^N, \tag{3.1}$$

where the partition function Z_V^N is

$$Z_V^N = \int_{\mathbb{U}(N)} \exp\left(N \operatorname{Tr} V\left(U^N, (U^N)^*, A_1^N, (A_1^N)^*, \ldots, A_p^N, (A_p^N)^*\right)\right) \mathrm{d}U^N. \tag{3.2}$$

We will evaluate all non-commutative polynomials at the matrices

$$U^N, (U^N)^*, A_1^N, (A_1^N)^*, \ldots, A_p^N, (A_p^N)^*$$

and will omit writing this explicitly in the sequel, e.g. writing $\mathrm{Tr}(V)$ to mean

$$\mathrm{Tr}\big(V(U^N,(U^N)^*,A_1^N,\dots,A_p^N)\big).$$

In Section 3.6, we will consider measures of the form

$$\frac{1}{Z_V^N}\exp(N\,\mathrm{Tr}\,V)\mathrm{d}U_1^N\cdots\mathrm{d}U_n^N,$$

where V is a non-commutative polynomial that depends on $U_1^N,\dots,U_n^N$, all independent and Haar-distributed.

We will assume the two following hypotheses.

Hypothesis 3.1.1. *For all $N\geq 1$ and for all $U_1,\dots,U_n\in\mathbb{U}(N)^n$, $\mathrm{Tr}\,V$ is real.*

Hypothesis 3.1.2. *Assume*

$$\sup_{N\geq 1}\sup_{1\leq i\leq p}\|A_i^N\|<\infty\,,$$

where $\|\cdot\|$ is the operator norm.

In most of the article, we will not assume Hypothesis 3.1.2, but rather assume Hypothesis 3.1.3:

Hypothesis 3.1.3. *For all $N\geq 1$ and for all $1\leq i\leq p$, $\|A_i^N\|\leq 1$, where $\|\cdot\|$ is the operator norm.*

This will prove convenient, and will not change the main result of this Chapter, Theorem 3.1.4 stated below. Stating Theorem 3.1.4 with Hypothesis 3.1.2 instead of 3.1.3 corresponds to rescaling the coefficients of the polynomial V.

Hypothesis 3.1.1 implies that the measure μ_V^N is a probability measure, and in particular that $Z_V^N\in(0,+\infty)$. We write the potential V as a sum of monomials q_i with complex coefficients z_i, $V=\sum_i z_iq_i$. Thus, we will sometimes consider the partition functions, cumulants, etc. as functions of $\boldsymbol{z}=(z_1,z_2,\dots)$. With this notation, the reality conditions is

$$\sum_i z_i\,\mathrm{Tr}(q_i)=\sum_i\overline{z_i}\,\mathrm{Tr}(q_i^*)\,.$$

Notice that for generic q_i's, $\mathrm{Tr}\,V$ might be real for only specific values of $\boldsymbol{z}$.

When considering the partition function with potential $V=tAU^NB(U^N)^*$, where $t\in\mathbb{C}$ and A,B are self-adjoint matrices, we recover the Harish-Chandra-Itzykson-Zuber (HCIZ) integral

$$Z_V^N=\int_{\mathbb{U}(N)}\exp\big(tN\,\mathrm{Tr}\big(AU^NB(U^N)^*\big)\big)\mathrm{d}U^N,$$

which was first studied by Harish-Chandra [Har57] and Itzykson and Zuber [IZ80], and whose asymptotics have been since investigated, see [ZZ03; GGN14; GN15; Nov20].

We will compute joint moments and cumulants (see Definition 2.2.5) of the random variables

$$\mathrm{Tr}(P_1),\dots,\mathrm{Tr}(P_l)\quad\text{ where } P_1,\dots,P_l \text{ are non-commutative polynomials}$$

under μ_V^N In [CGM09], the first-order asymptotics of partition functions was studied. In [GN15], it has been shown that the joint cumulants admit an asymptotic expansion as $N\to\infty$, when the coefficients of the potential V are small enough.

The goal of this article is to give a combinatorial interpretation of the coefficients of this expansion. We show that unitary matrix integrals enumerate a particular family of maps, which we call maps of unitary type. They are introduced in Section 3.3.2, Definition 3.3.7. This interpretation links the Dyson-Schwinger equation, which is satisfied by sums of maps of unitary type, and the Weingarten calculus. The Weingarten calculus was studied first by Weingarten [Wei78], and then by [Sam80], whose results were rediscovered and expanded by Collins [Col03], and Collins and Śniady [CŚ06]. See [CMN22] for a review.

Expansions in terms of combinatorial objects have already been introduced for unitary matrices. For instance, in the case of the HCIZ integral, expansions for the free energy using double Hurwitz numbers are computed in [GGN11]. In [CGM09], the leading order of the expansion of unitary integrals is expressed in terms of maps with "dotted edges". However, to our knowledge, no interpretation of these expansions using maps has been obtained at all orders for the unitary integrals we consider. As an interesting particular case, when considering alternated polynomials (see Definition 3.3.33), the combinatorics of maps of unitary type is related to triple Hurwitz numbers.

In the case of the GUE, integrals of random matrices and enumeration of maps are related by Wick's formula, recalled in Section 2.5.1. In the case of unitary matrices, Wick's formula is replaced by Weingarten's formula. In Section 3.2, we express joint moments of random variables $\mathrm{Tr}(P_i)$, for non-commutative polynomials P_i, using Weingarten's formula. In the case where the potential $V = 0$, we can express such moments as weighted sums of permutations. In Section 3.3, we recall a few notions on maps and introduce the maps of unitary type, which are our main combinatorial tools. This allows us to deduce a topological expansion for the joint cumulants in the case of no potential (i.e. $V = 0$). To address the general case $V \neq 0$, we introduce generating series of maps of unitary type of the form

$$\mathcal{M}_{V,l}^{(g),N}(P_1, \dots, P_l) = \sum_{\boldsymbol{n} \in \mathbf{N}^k} \frac{\boldsymbol{z}^{\boldsymbol{n}}}{\boldsymbol{n}!} \times \sum w_N(\mathfrak{m}, \boldsymbol{n}, V, P_1, \dots, P_l),$$

where the second sum is on a set of connected maps $\mathfrak{m}$ of unitary type (see Definition 3.3.7) of genus g which depends on $V, P_1, \dots, P_l, \boldsymbol{n}$. The term $w_N(\mathfrak{m}, \boldsymbol{n}, V, P_1, \dots, P_l)$ is a weight which depends on the size N, $\mathfrak{m}$, $\boldsymbol{n}$ and on the polynomials $V, P_1, \dots, P_l$. See Definition 3.3.32.

In Section 3.5, we describe a decomposition of maps of unitary type, which can be interpreted as a cutting procedure. It allows us to deduce induction relations – similar to the topological recursion of Chekhov, Eynard and Orantin, see [CE06; EO08] – on weighted sums $\mathcal{M}_{V,l}^{(g),N}$ of maps of unitary type of a given genus g. This decomposition is reminiscent of a procedure introduced by Tutte [Tut68]. In Section 3.6, we extend the results obtained so far to the case of integrals over several independent random unitary matrices $U_1^N, \dots, U_n^N$.

It turns out that the induction relations obtained in Section 3.5 are related to the Dyson-Schwinger lattice. The Dyson-Schwinger lattice (see [GN15]) is a family of equations relating cumulants together, which generalize the Dyson-Schwinger equation (see Equation (3.30)). This equation admits under some hypotheses a unique solution [CGM09]. Furthermore, in [GN15], the Dyson-Schwinger lattice has been used to establish the existence of an asymptotic expansion of the cumulants, when $N \to \infty$. Let us assume Hypotheses 3.1.1 and 3.1.3, and that the joint law of the matrices A_i^N, tr admits an asymptotic expansion as $N \to \infty$. For all h, we have an asymptotic expansion for the renormalized joint cumulants $N^{l-2}\mathcal{W}_{V,l}^N(P_1, \dots, P_l)$ (introduced in Definition 3.2.1) when the coefficients of the potential V are small enough

$$N^{l-2}\mathcal{W}_{V,l}^N(P_1, \dots, P_l) = \sum_{g=0}^{h} \frac{\tau_{l,g}^V(P_1, \dots, P_l)}{N^{2g}} + o(N^{-2h}), \tag{3.3}$$

where the coefficients $\tau_{l,g}^V(P_1, \dots, P_l)$ are uniquely defined by some induction relations.

In Section 3.7, we use the same techniques to express the terms of this expansion in terms of maps of unitary type. We thus obtain a topological expansion: the coefficient of $\frac{1}{N^{2g}}$ in the expansion is a generating series of weighted unitary type maps of genus g.

We thus improve on the result of [GN15, Theorem 25] by relaxing the hypotheses, showing that the convergence is uniform in g and l, and by giving a combinatorial interpretation to the coefficients $\tau_{l,g}^V(P_1, \dots, P_l)$. Our result is the following.

Theorem 3.1.4. *Assume that for all $N \geq 1$, $\mathrm{Tr}(V)$ is real for all $U_1, \dots, U_n \in \mathbb{U}(N)^n$ and that*

$$\sup_{N \geq 1} \sup_{1 \leq i \leq p} \|A_i^N\| < \infty.$$

There exists $\epsilon > 0$ *such that if*

$$\|\boldsymbol{z}\|_\infty < \epsilon,$$

then for all $l \geq 1$, $g \geq 0$, *and* $\boldsymbol{P} = (P_1, \dots, P_l)$, *we have the asymptotic expansion as* $N \to \infty$

$$N^{l-2}\mathcal{W}_{V,l}^N(P_1, \dots, P_l) = \sum_{h=0}^{g} \frac{1}{N^{2h}} \mathcal{M}_{V,l}^{(h),N}(P_1, \dots, P_l) + \mathcal{O}\big(N^{-2g-2}\big).$$

Notice that we do not require the trace Tr to have an asymptotic expansion as in [GN15, Theorem 25].

An interesting particular case described in Section 3.3.6 is when all the polynomial involved are **alternated**, see Definition 3.3.33, that is if they can be written as

$$P = B_1^N U^N C_1^N (U^N)^* \cdots B_m^N U^N C_m^N (U^N)^*,$$

where B_i^N and C_i^N for $i = 1, \dots, m$ are square $N \times N$ matrices. This is the case of the HCIZ integral in particular. In that case, our sums of maps are related to the triple monotone Hurwitz numbers, which count some ramified coverings of the sphere with at most three non-simple ramification points. We thus generalize the link between the (double) monotone Hurwitz numbers and the HCIZ integral, which had already been studied in [GGN14]. See also [CGL23] for a study of the HCIZ integral in the tensor setting.

In Section 3.2, we give definitions and recall important consequences of the Weingarten calculus. In Section 3.3, we introduce the maps of unitary types and show that they describe the topological expansion of cumulants with respect to the Haar measure. When the polynomial are alternated, these maps are related to the triple Hurwitz numbers. In Section 3.5, we give a decomposition of maps of unitary type and deduce induction relations on sums of maps of a given genus and with prescribed vertices, in the spirit of the work of Tutte [Tut68]. In Section 3.7, we study the Dyson-Schwinger equation and give the proof of the main result.

3.2 Weingarten calculus

In this section, we first give a few definitions and introduce notation pertaining to moments and cumulants of traces of random matrices. Then, we give a short review of the Weingarten calculus. This allows us to give expression for the expectation of a product of traces of monomials in the matrices $U^N, (U^N)^*, A_i^N, (A^*)^N$.

3.2.1 Moments and cumulants

Let us consider $l \geq 1$ non-commutative polynomials $P_1, P_2, \dots, P_l$ in the variables u, u^{-1}, and a_i, a_i^* for $1 \leq i \leq p$, with $p \in \mathbf{N}$. We define the involution $*$ such that $u^* = u^{-1}$, for $1 \leq i \leq p$, $(a_i)^* = a_i^*$, and for any letters $X_1, \dots, X_k$ in $\{u, u^*, a_i, a_i^* \colon 1 \leq i \leq p\}$ and $z \in \mathbb{C}$, we have $(zX_1 \cdots X_k)^* = z^* X_k^* \cdots X_1^*$. We denote the unital $*$-algebra generated by such polynomials by

$$\mathcal{A} = \mathbb{C}\langle u, u^{-1}, a_i, a_i^*; 1 \leq i \leq p\rangle.$$

The unital $*$-algebra generated by the non-commutative polynomials in $a_1, a_1^*, \dots, a_p, a_p^*$ only is denoted by $\mathcal{B}$. Notice that there is no relation between the formal variables u and u^{-1}, or a_i and a_i^* for $i \in \mathbf{N}^*$ (except for those involving $*$).

In this article, we study the random variables $\mathrm{Tr}(P_1), \dots, \mathrm{Tr}(P_l)$, seen as functions of U^N, under the measure μ_V^N (see (3.1)). We will be interested in computing the joint moments and cumulants of these random variables.

Definition 3.2.1. *For $(P_1,\dots,P_l)\in\mathcal{A}^l$, we write the joint moments of the traces of P_i's under μ_V^N as*

$$\alpha_{V,l}^N(P_1,\dots,P_l)=\mathbb{E}\left[\operatorname{Tr}(P_1),\dots,\operatorname{Tr}(P_l)\right]=\int_{\mathbb{U}(N)}\operatorname{Tr}(P_1)\cdots\operatorname{Tr}(P_l)\mathrm{d}\mu_V^N\,.$$

We write the joint cumulants under μ_N^V as

$$\mathcal{W}_{V,l}^N(P_1,\dots,P_l)=\kappa_l(\operatorname{Tr}(P_1),\dots,\operatorname{Tr}(P_l)),$$

and introduce the renormalized cumulants

$$\tilde{\mathcal{W}}_{V,l}^N(P_1,\dots,P_l)=N^{l-2}\kappa_l(\operatorname{Tr}(P_1),\dots,\operatorname{Tr}(P_l)).$$

In Section 3.7, we will discuss an asymptotic expansion (as $N\to\infty$) for the joint cumulants. For now, we study the moments for N fixed. When $V=0$, we can compute directly the moments using Weingarten's formula, see Subsection 3.2.2. When $V\neq 0$, we can compute the cumulants using the free energy F_V^N defined in terms of the partition function Z_V^N. Recall that $V=\sum_{i=1}^k z_iq_i$ is the potential, a sum of k polynomials $q_1,\dots,q_k\in\mathcal{A}$ with complex coefficients $z_1,\dots,z_k$. Note that V does not depend on N. We have

$$Z_V^N=\int_{\mathbb{U}(N)}\exp(N\operatorname{Tr}(V))\mathrm{d}U^N,$$

and we define the free energy as

$$F_V^N=\frac{1}{N^2}\ln Z_V^N. \tag{3.4}$$

The free energy is always well defined when $\operatorname{Tr}V$ is real.

In the expression of the partition function, we can develop the exponential as a series and exchange the sum and the integral:

$$\begin{aligned}Z_V^N&=\int_{\mathbb{U}(N)}\sum_{n_1,\dots,n_k\geq 0}\prod_{i=1}^k\frac{(Nz_i\operatorname{Tr}(q_i))^{n_i}}{n_i!}\mathrm{d}U^N\\&=\sum_{n_1,\dots,n_k\geq 0}\prod_{i=1}^k\frac{(Nz_i)^{n_i}}{n_i!}\int_{\mathbb{U}(N)}\operatorname{Tr}(q_1)^{n_1}\cdots\operatorname{Tr}(q_k)^{n_k}\mathrm{d}U^N\,.\end{aligned}$$

In the second line, we used Hypothesis 3.1.3, which implies that $|\operatorname{Tr}(q_i)|\leq N$, and the fact that we are integrating with respect to the Haar measure on the compact group $\mathbb{U}(N)$ to exchange the sum and the integral. Notice that this expression is valid for all $\boldsymbol{z}$, even if $\operatorname{Tr}V$ is not real.

We introduce the notation $\boldsymbol{z}=(z_1,\dots,z_k)$, and for $\boldsymbol{n}=(n_1,\dots,n_k)\in\mathbf{N}^k$, $\boldsymbol{z}^{\boldsymbol{n}}=\prod_{i=1}^k z_i^{n_i}$ and $\boldsymbol{n}!=\prod_{i=1}^k n_i!$. Then,

$$Z_V^N=\sum_{n\geq 0}N^n\sum_{\substack{\boldsymbol{n}\in\mathbf{N}^k\\ n_1+\cdots+n_k=n}}\frac{\boldsymbol{z}^{\boldsymbol{n}}}{\boldsymbol{n}!}\alpha_{0,n}^N(\underbrace{q_1,\dots,q_1}_{n_1\text{times}},\dots,\underbrace{q_k,\dots,q_k}_{n_k\text{times}}),$$

and therefore the partition function is a generating series of the moments with respect to the Haar measure (i.e. with $V=0$).

Similarly, the free energy is a generating series of the renormalized cumulants for $V=0$ (see [Bon15, Theorem 1.3.3, 4.])

$$\begin{aligned}F_V^N&=\sum_{n\geq 1}\sum_{\substack{\boldsymbol{n}\in\mathbf{N}^k\\ n_1+\cdots+n_k=n}}\frac{(N\boldsymbol{z})^{\boldsymbol{n}}}{\boldsymbol{n}!}\frac{1}{N^2}\mathcal{W}_{0,n}^N(\underbrace{q_1,\dots,q_1}_{n_1\text{ times}},\dots,\underbrace{q_k,\dots,q_k}_{n_k\text{ times}})\\&=\sum_{n\geq 1}\sum_{\substack{\boldsymbol{n}\in\mathbf{N}^k\\ n_1+\cdots+n_k=n}}\frac{\boldsymbol{z}^{\boldsymbol{n}}}{\boldsymbol{n}!}\tilde{\mathcal{W}}_{0,n}^N(\underbrace{q_1,\dots,q_1}_{n_1\text{ times}},\dots,\underbrace{q_k,\dots,q_k}_{n_k\text{ times}})\end{aligned}$$

Notice that the free energy a priori exists only for $\boldsymbol{z}$ sufficiently small. Indeed, Z_V^N is defined for all $\boldsymbol{z}$ but is nonzero on a open neighborhood of 0 which depends on N. In particular, the radius of convergence of F_V^N a priori depends on N.

Notice that by modifying the potential V and differentiating, we have

$$\frac{\partial}{\partial t}\Big|_{t=0} F^N_{V+tP} = \frac{1}{N}\int_{\mathbb{U}(N)} \mathrm{Tr}(P)\mathrm{d}\mu_V^N(U^N) = \frac{1}{N}\alpha^N_{V,1}(P) = \tilde{\mathcal{W}}^N_{V,1}(P).$$

In general, we can prove by induction the following lemma, which is a consequence of the definition of cumulants (2.5) in terms of their generating function.

Lemma 3.2.2. *The renormalized joint cumulants are given by*

$$\tilde{\mathcal{W}}^N_{V,l}(P_1,\dots,P_l) = \frac{\partial^l}{\partial t_1\partial t_2\cdots\partial t_l}\Big|_{t_1=\dots=t_l=0} F^N_{V+\sum_i t_iP_i}.$$

Lemma 3.2.2 implies that for a fixed N, there exists a neighborhood $U_0 \in \mathbb{C}^k$ of 0 such that for $\boldsymbol{z} \in U_0$,

$$\tilde{\mathcal{W}}^N_{V,l}(P_1,\dots,P_l) = \sum_{n\geq 0}\sum_{\substack{\boldsymbol{n}\in\mathbf{N}^k\\ n_1+\dots+n_k=n}} \frac{\boldsymbol{z}^{\boldsymbol{n}}}{\boldsymbol{n}!}\tilde{\mathcal{W}}^N_{0,n}(\underbrace{q_1,\dots,q_1}_{n_1 \text{ times}},\dots,\underbrace{q_k,\dots,q_k}_{n_k\text{ times}},P_1,\dots,P_l). \tag{3.5}$$

In the next subsections, we compute the moments with respect to the Haar measure. From these moments and Definition 2.2.5, we can compute the cumulants with respect to the Haar measure. The expression (3.5) motivates the introduction in Section 3.3.5 of a formal sum. The first terms of this sum are shown to give the asymptotic expansion of the cumulants in Theorem 3.1.4.

3.2.2 The Weingarten formula

To compute the moments with respect to the Haar measure, the key tool is Weingarten's formula, first obtained in [Wei78], which expresses the average of coefficients of a unitary matrix in terms of the Weingarten function defined below (Definition 3.2.3). See [CMN22] for a review on the Weingarten calculus.

Definition 3.2.3. *Let $q \leq N$ be an integer. The **Weingarten function** $\mathrm{Wg}_N\colon \mathfrak{S}_q \to \mathbb{C}$ is defined for all $\pi \in \mathfrak{S}_q$ by*

$$\mathrm{Wg}_N(\pi) = \int_{\mathbb{U}(N)} (U^N)_{11}\cdots(U^N)_{qq}\overline{(U^N)_{1\pi(1)}\cdots(U^N)_{q\pi(q)}}\mathrm{d}U^N.$$

This function can also be defined for all $q \in \mathbf{N}^*$ using characters of the symmetric group (see [CŚ06]). The invariance of the Haar measure by multiplication by permutation matrices implies that the Weingarten function is invariant by conjugation, i.e. for all $\sigma,\pi \in \mathfrak{S}_q$ we have

$$\mathrm{Wg}_N(\sigma\pi\sigma^{-1}) = \mathrm{Wg}_N(\pi).$$

With our definition of the Weingarten function, Weingarten's formula is valid in the case $q \leq N$. It actually holds for all $q \geq 1$ with an appropriate definition of the Weingarten function.

Theorem 3.2.4. *(Weingarten's formula, see [Col03] and [CŚ06]) Let U^N be a Haar-distributed unitary matrix of size $N\times N$ and $\boldsymbol{i} = (i_1,i_2,\dots,i_q), \boldsymbol{j} = (j_1,j_2,\dots,j_q), \boldsymbol{i'} = (i'_1,i'_2,\dots,i'_{q'})$ and $\boldsymbol{j} = (j'_1,j'_2,\dots,j'_{q'})$ be elements of $[N]^q$ or $[N]^{q'}$ for $q,q'\geq 1$.*

$$\int_{\mathbb{U}(N)} (U^N)_{i_1j_1}\cdots(U^N)_{i_qj_q}\overline{(U^N)_{i'_1j'_1}\cdots(U^N)_{i'_{q'}j'_{q'}}}\mathrm{d}U^N \\ = \delta_{q,q'}\sum_{\rho,\sigma\in\mathfrak{S}_q}\prod_{k=1}^q \delta_{i_k,i'_{\sigma(k)}}\delta_{j_k,j'_{\rho(k)}}\mathrm{Wg}_N(\sigma\rho^{-1}). \tag{3.6}$$

Before giving the expression for the moments $\alpha^N_{0,l}(P_1,\ldots,P_l)$ with respect to the Haar measure, let us make a simplifying assumptions on our polynomials P_i.

We introduce the set $\mathcal{Y}$ of words in the letters $a_1, a_1^*, \ldots, a_p, a_p^*$. We assume that for all i, P_i can be written uniquely as

$$M_{i,1}u^{\epsilon_{i,1}}M_{i,2}u^{\epsilon_{i,2}}\cdots M_{i,d_i}u^{\epsilon_{i,d_i}}\,, \tag{3.7}$$

where $M_{i,j}$ is either the empty word or an element of $\mathcal{Y}$, $d_i \geq 1$, and $\boldsymbol{\epsilon_i} = (\epsilon_{i,1},\ldots,\epsilon_{i,d_i}) \in \{\pm 1\}^{d_i}$. We write $\mathcal{X}$ the set of such monomials. We have $\mathcal{Y} \subset \mathcal{X}$. Notice that $\mathcal{A}$ is generated by the elements of $\mathcal{X}$ up to cyclic permutation of the factors in a monomial.

The integer d_i, that we will sometime write $\deg P_i$, is the **degree** of the monomial P_i. Notice that there is no relation between the formal variables, in particular between u and u^{-1} (except for those involving $*$). Therefore, the degree of (3.7) is defined by counting the total number of letter u or u^* in a word. In particular, $\deg(uu^{-1}) = 2$.

Definition 3.2.5. *With $(P_1,\ldots,P_l) \in \mathcal{X}^l$, and with the notation (3.7), we set*

- $\boldsymbol{P} = (P_1,\ldots,P_l)$,
- $\boldsymbol{M_P} = (M_i)_{i\in[\sum_i \deg P_i]} = (M_{1,1},\ldots,M_{1,d_1},\ldots,M_{l,1},\ldots,M_{l,d_l})$,
- $\boldsymbol{\epsilon_P} = (\epsilon(i))_{i\in[\sum_i \deg P_i]} = (\epsilon_{1,1},\ldots,\epsilon_{1,d_1},\ldots,\epsilon_{l,1},\ldots,\epsilon_{l,d_l})$.

Notice that we change the indices of the monomials $M_{i,j}$ and of $\boldsymbol{\epsilon}_{i,j}$, by setting for all $1 \leq i \leq k, 1 \leq j \leq d_i$, $M_{d_1+\cdots+d_{i-1}+j} = M_{i,j}$ and $\epsilon(d_1+\cdots d_{i-1}+j) = \epsilon_{i,j}$.

We set $\deg \boldsymbol{P} = \sum_i \deg P_i$.

Furthermore, we define the permutation

$$\gamma_{\boldsymbol{P}} = (1\,\ldots\,d_1)(d_1+1\,\ldots\,d_2)\cdots(d_{l-1}+1\,\ldots\,d_l)\,. \tag{3.8}$$

In the sequel, we shall consider $\boldsymbol{\epsilon_P}$ as a function, but sometime using vector notation for convenience. In particular, we consider the sets. $\boldsymbol{\epsilon_P}^{-1}(+1) = \{i \in [\deg \boldsymbol{P}] \colon \boldsymbol{\epsilon_P}(i) = +1\}$ and $\boldsymbol{\epsilon_P}^{-1}(-1) = \{i \in [\deg \boldsymbol{P}] \colon \boldsymbol{\epsilon_P}(i) = -1\}$.

Remark 3.2.6. The permutation $\gamma_{\boldsymbol{P}}$ defined by (3.8) gives a choice of labelling for the letters u and u^* in the monomials we consider. Notice that this choice is arbitrary. The cycle notation is convenient here as we are interested in traces of such monomials. The ordering of the letters in the words needs only to be specified up to cyclic permutation.

For any permutation $\sigma \in \mathfrak{S}_{\deg \boldsymbol{P}}$, we can replace $\gamma_{\boldsymbol{P}}, \boldsymbol{M_P} = (M_i), \boldsymbol{\epsilon_P} = (\epsilon(i))$ by

$$\gamma' = \sigma^{-1}\gamma_{\boldsymbol{P}}\sigma, \quad \boldsymbol{M'} = (M'_i) = (M_{\sigma(i)}), \quad \text{and} \quad \boldsymbol{\epsilon}' = (\epsilon'(i)) = (\epsilon(\sigma(i))).$$

This new data describes the same polynomials. By this we means that if we write $\gamma' = c'_1\cdots c'_l$ the decomposition in disjoint cycles of γ', we have

$$\prod_{i=1}^{l} \mathrm{Tr}(P_i) = \prod_{i=1}^{l} \mathrm{Tr}\left(\overrightarrow{\prod}_{j\in c_i} M_j u^{\boldsymbol{\epsilon}'(j)}\right)\,.$$

Notice that the non-commutative product is only defined up to the cyclic permutation of the factors. The cyclic property of the trace ensures that the quantity on the right-hand side is well defined.

We can assume all the polynomials are of the form (3.7) without loss of generality as $\alpha^N_{V,l}$ is multilinear and satisfies the trace property

$$\alpha^N_{V,l}(P_1,\ldots,P_{i-1},P_iQ,P_{i+1},\ldots,P_l) = \alpha^N_{V,l}(P_1,\ldots,P_{i-1},QP_i,P_{i+1},\ldots,P_l)\,,$$

as $\mathrm{Tr}(P_iQ) = \mathrm{Tr}(QP_i)$. Furthermore, if there exists i such that P_i contains no letter u nor u^{-1}, we can factor the term $\mathrm{Tr}(P_i)$ out of the moment.

The formula for the moments with respect to the Haar measure involves permutations belonging to the set $\mathfrak{S}^{(\epsilon)}(I) \subset \mathfrak{S}(I)$ of permutations (introduced in [MŚS07]), for $\epsilon \in \{\pm 1\}^I$.

Definition 3.2.7. *Let $\epsilon \in \{\pm 1\}^I$. The set $\mathfrak{S}^{(\epsilon)}(I) \subset \mathfrak{S}(I)$ is the set of permutations $\pi \in \mathfrak{S}(I)$ such that*

$$\pi\left(\epsilon^{-1}(+1)\right) = \epsilon^{-1}(-1)\,.$$

Furthermore, we define $\pi^{(\epsilon)} = \pi^2|_{\epsilon^{-1}(+1)} \in \mathfrak{S}(\epsilon^{-1}(+1))$.

Notice that the set $\mathfrak{S}^{(\epsilon)}(I)$ is empty if $|\epsilon^{-1}(+1)| \neq |\epsilon^{-1}(-1)|$.

Example 3.2.8. For instance, if $\epsilon = (+1,+1,-1,+1,-1,-1)$, then $\pi = (1\,3\,4\,6)(2\,5) \in \mathfrak{S}_6^{(\epsilon)}$, and $\pi^{(\epsilon)} = (1\,4)(2)$.

The notation of Definitions 3.2.5 and 3.2.7 allows us to express the moments in a compact way.

Proposition 3.2.9 ([MŚS07, Proposition 3.4])**.** *Let $\boldsymbol{P} = (P_1,\ldots,P_l) \in \mathcal{X}^l$. We have*

$$\alpha_{0,l}^N(\boldsymbol{P}) = \alpha_{0,l}^N(P_1,\ldots,P_l) = \sum_{\pi\in\mathfrak{S}_{\deg \boldsymbol{P}}^{(\epsilon_{\boldsymbol{P}})}} \mathrm{Tr}_{\gamma_{\boldsymbol{P}}\pi^{-1}}(\boldsymbol{M}_{\boldsymbol{P}})\mathrm{Wg}_N(\pi^{(\epsilon_{\boldsymbol{P}})}). \tag{3.9}$$

3.2.3 Expansion of the Weingarten function

We wish to express the moments and cumulants uniquely in terms of combinatorial objects and traces. To this end, we now present a result of Novak [Nov10] expressing the Weingarten function in terms of walks on the Cayley graph of $\mathfrak{S}_n$ generated by the transpositions.

Definition 3.2.10. *The* ***value*** *of a transposition $\tau = (i\,j) \in \mathfrak{S}(I)$, where I is a finite subset of $\mathbf{N}^*$, is* $\mathrm{val}(\tau) = \max\{i,j\}$.

Definition 3.2.11. *Let ρ and σ be in $\mathfrak{S}(I)$, with I a finite subset of $\mathbf{N}^*$.*

A ***(weakly) monotone walk with*** r ***steps on*** $\mathfrak{S}(I)$ *from ρ to σ is a tuple $(\tau_1,\ldots,\tau_r)$ of transpositions of $\mathfrak{S}(I)$ such that*

- $\tau_r\cdots\tau_1\rho = \sigma$, *and*
- $\mathrm{val}(\tau_1) \leq \cdots \leq \mathrm{val}(\tau_r)$.

We denote the set of such walks by $\overrightarrow{\mathcal{W}}^r(\rho,\sigma)$, and we define $\overrightarrow{w}^r(\rho,\sigma)$ as the cardinality of the set $\overrightarrow{\mathcal{W}}^r(\rho,\sigma)$.

In this Definition, we use the arrow notation $\overrightarrow{w}$ and $\overrightarrow{\mathcal{W}}$ to emphasize the monotonicity property, as in [GGN14].

Proposition 3.2.12 ([Nov10, Theorem 3.1])**.** *Let $\pi \in \mathfrak{S}_q$ with $N \geq q$. We have*

$$\mathrm{Wg}_N(\pi) = \sum_{r\geq 0} \frac{(-1)^r}{N^{r+q}} \overrightarrow{w}^r(\mathrm{Id},\pi),$$

and the series is absolutely convergent.

Propositions 3.2.12 and 3.2.9 imply the following result (recall notation from Definition 3.2.5).

Corollary 3.2.13. *Let $N \geq 1$ be an integer, $\boldsymbol{P} = (P_1,\ldots,P_l) \in \mathcal{X}^l$ with $m = \deg \boldsymbol{P}/2 \leq N$. The moments admits the expansion*

$$\alpha_{0,l}^N(P_1,\ldots,P_l) = \sum_{r\geq 0} \frac{(-1)^r}{N^{r+m}} \sum_{\pi\in\mathfrak{S}_{2m}^{(\epsilon_P)}} \mathrm{Tr}_{\gamma_{\boldsymbol{P}}\pi^{-1}}(\boldsymbol{M}_{\boldsymbol{P}})\overrightarrow{w}^r(\mathrm{Id},\pi^{(\epsilon_P)}).$$

Moreover, the series is absolutely convergent, uniformly on $\boldsymbol{M}_{\boldsymbol{P}}$.

Notice that if $\deg P$ is odd, there are a different number of occurrences of u and of u^*, and such moments are 0.

Proof. Starting from the expression for the moment of Proposition 3.2.9, we use the expansion for the Weingarten function of Proposition 3.2.12. Notice that this second result can only be used if $m = \deg \boldsymbol{P}/2 \leq N$, where m is the total number of letter u in the monomials $P_1, \dots, P_l$. One of the sums is finite, we can exchange the sums and get the wanted expression.

Finally, as the matrices (A_i) have their operator norm bounded by 1, we can crudely bound the trace by

$$|\operatorname{Tr}_{\gamma_{\boldsymbol{P}}\pi^{-1}}(\boldsymbol{M_P})| \leq N^{c(\gamma_{\boldsymbol{P}}\pi^{-1})} \leq N^{2m}\,.$$

This implies that the convergence is uniform in $\boldsymbol{M_P}$. □

3.3 Oriented maps and maps of unitary type

In this section, we introduce combinatorial objects, the so-called maps of unitary type, that will be convenient to express the moments $\alpha_{0,l}^N$, and then the cumulants $\mathcal{W}_{0,l}^N$. These maps are particular cases of the maps appearing in the Gaussian case.

3.3.1 Oriented maps

We defined maps in Section 2.4.1. Here, we consider maps whose edges are oriented.

Remark 3.3.1. In this Chapter, we depart in two ways from the discussion of Section 2.4.1:

- The embedded graphs, and hence the maps, are in general **disconnected**. We will specify it when the maps we consider are connected.
- Given a half-edge labelled map $\mathfrak{m}$, each cycle of the permutation $\varphi_{\mathfrak{m}}$ corresponds to a face explored in the **counterclockwise** orientation, and not the clockwise orientation. In particular, the different permutations describing $\mathfrak{m}$ are related by

$$\varphi_{\mathfrak{m}} = \sigma_{\mathfrak{m}}\alpha_{\mathfrak{m}}.$$

Definition 3.3.2. *A face will be said to be* ***incident*** *to a vertex or an edge if the vertex or the edge belongs to the boundary of the face.*

It will be convenient to regard each edge of a map as being made of two half-edges. When going from the vertex of the half-edge to the other end of the half-edge (connected to another half-edge), we can distinguish a left side and a right side (see Figure 3.1). Notice that the left and right side are defined relative to the position of the incident vertex, and does not depend on the possible orientation.

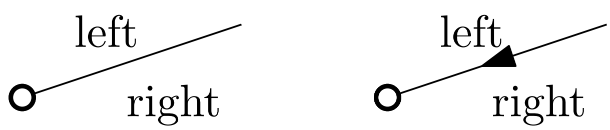


Figure 3.1: The left and right side of a half-edge, and of an ingoing half-edge.

We label the half-edges of a map $\mathfrak{m}$ from 1 to $2m$, where m is the number of edges of $\mathfrak{m}$. By convention, we write each label at the left of its half-edge. See Figure 3.2. In an oriented map with labelled half-edges, the edges can be represented as an ordered pair of two half-edges. The first half-edge is connected to the first vertex of the edge and is said to be **outgoing**. The second half-edge is connected to the second vertex of the edge and is said to be **ingoing**.

For an oriented map, we must also describe the orientation of each half-edge.

Definition 3.3.3. *Let $\mathfrak{m}$ be an oriented map with $2m$ labelled half-edges. We define the function $\boldsymbol{\epsilon}_{\mathfrak{m}} \in \{\pm 1\}^{[2m]}$ as follows. For all $i \in [2m]$, we set $\epsilon(i) = +1$ if the half-edge labelled i is outgoing and $\epsilon(i) = -1$ if the half-edge labelled i is ingoing.*

Such an $\boldsymbol{\epsilon}$ belongs to the set $\mathcal{E}_{2m} = \{\boldsymbol{\epsilon} \in \{\pm 1\}^{2m} \colon \sum_{i=1}^{2m} \epsilon(i) = 0\}$.

In the case of an oriented map, α is in the set $\mathcal{I}_{2m}^{(\epsilon)}$ defined by

$$\mathcal{I}_{2m}^{(\epsilon)} := \{\alpha \in \mathcal{I}_{2m} \colon \forall i \in [2m], \epsilon(\alpha(i)) = -\epsilon(i)\}\,. \tag{3.10}$$

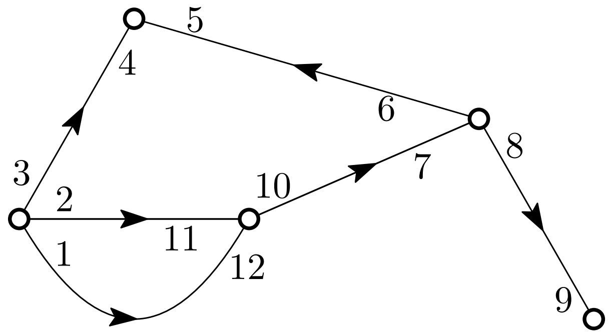


Figure 3.2: A map with labelled half-edges.

Example 3.3.4. The oriented map $\mathfrak{m}$ of Figure 3.2 is described by

$$\begin{aligned}
\sigma_{\mathfrak{m}} &= (1\,3\,2)(4\,5)(6\,8\,7)(9)(10\,12\,11)\\
\alpha_{\mathfrak{m}} &= (1\,12)(2\,11)(3\,4)(5\,6)(7\,10)(8\,9)\\
\varphi_{\mathfrak{m}} &= (1\,11)(2\,10\,6\,4)(3\,5\,8\,9\,7\,12)\\
\boldsymbol{\epsilon}_{\mathfrak{m}} &= (+1,+1,+1,-1,-1,+1,-1,+1,-1,+1,-1,-1)\,.
\end{aligned}$$

We now state the counterpart of Theorem 2.4.12 for oriented maps.

Theorem 3.3.5. *Let $m \geq 1$, $\sigma \in \mathfrak{S}_{2m}, \boldsymbol{\epsilon} \in \mathcal{E}_{2m}$ and $\mathfrak{C}(m, \boldsymbol{\epsilon}, \sigma)$ be the set of oriented maps with $2m$ labelled half-edge $\mathfrak{m}$ such that $\sigma_{\mathfrak{m}} = \sigma$ and $\boldsymbol{\epsilon}_{\mathfrak{m}} = \boldsymbol{\epsilon}$. Then,*

$$\begin{aligned}
\mathfrak{C}(m, \epsilon, \sigma) &\to \mathcal{I}_{2m}^{(\boldsymbol{\epsilon})}\\
\mathfrak{m} &\mapsto \alpha_{\mathfrak{m}},
\end{aligned}$$

is a bijection.

3.3.2 Maps of unitary type

We have just seen how to describe a map with permutations. We now define a particular type of map, which we call map of unitary type, whose edge structure is described by a permutation $\pi \in \mathfrak{S}_{2m}^{(\epsilon)}$ for some ϵ and $m \geq 1$ and a monotone walk $(\tau_1, \ldots, \tau_r) \in \overrightarrow{\mathcal{W}}^r(\mathrm{Id}, \pi^{(\epsilon)})$.

Definition 3.3.6. *A vertex in an oriented map will be said to be **alternated** if when going around this vertex the half-edges connected to it are alternatively ingoing and outgoing.*

Definition 3.3.7. *Let I be a finite subset of $\mathbf{N}^*$ and $r \in \mathbf{N}$. A **map of unitary type** with labels in I with r black vertices is an oriented map with vertices colored in white or black such that*

1. *there are r black vertices, which are alternated of degree 4 and numbered from 1 to r;*

2. *there are $|I|$ half-edges that are connected to white vertices. We call these half-edges **white half-edges**. Each element of I labels exactly one white half-edge;*

3. *if an oriented edge connects the black vertex numbered k to the black vertex numbered l, with the orientation from k to l, then $k < l$.*

See Figure 3.3 for an example.

Remark 3.3.8. There is a correspondence between a tuple $\boldsymbol{P} = (P_1, \ldots, P_l)$ of monomials and a family of maps of unitary type. The number of white vertices is l, each of them corresponds to a monomial. The white outgoing half-edges correspond to occurrences of u, the white ingoing half-edges correspond to occurrences of u^*. The black vertices correspond to steps in a walk as defined in Definition 3.2.11. Note however that the monotonicity condition of the walk correspond to the increasing condition defined in Definition 3.3.14. This link will be described in more details in Section 3.3.4.

Remark 3.3.9. The map has oriented edges so there are as many ingoing as outgoing half-edges, of any color. The black vertices are alternated and of degree 4 so there are as many ingoing black half-edges as black outgoing half-edges. Thus, there are as many white ingoing half-edges as white outgoing half-edges.

Remark 3.3.10. Notice that condition 3. in Definition 3.3.7 implies that each face is incident to at least one white vertex. Indeed, if it were not the case, there would be a face incident to only black vertices, numbered $n_1 < n_2 < \ldots < n_k$, with $n_k < n_1$, a contradiction.

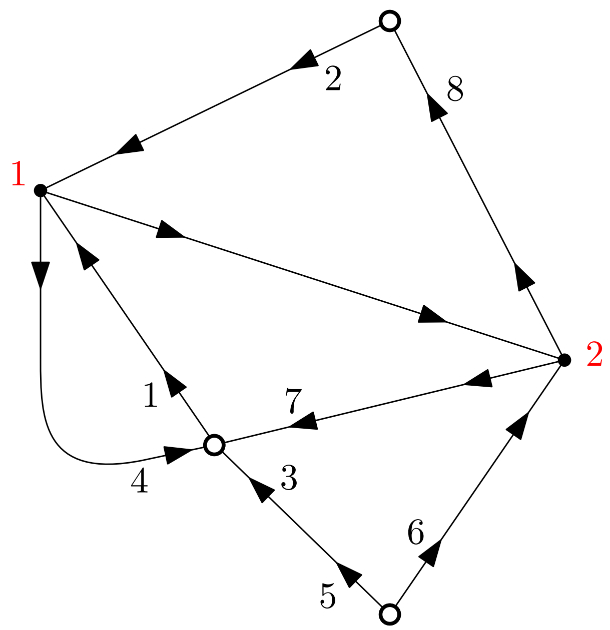


Figure 3.3: A unitary type map. The numbers in red (1 near the black vertex the left, 2 near the black vertex on the right) are the numbers of the black vertices, the labels in black are the labels of the white half-edges.

Remark 3.3.11. The maps of unitary type are very similar to the maps introduced in [CGM09] to describe the leading term in the asymptotics of the cumulants when $N \to \infty$. In fact, the two kinds of maps in genus 0 are related by a surgery that transforms black vertices of unitary maps into "dotted edges" of the maps from [CGM09]. Here, we consider the non-planar cases as well.

We denote by $w_k(\mathfrak{m})$ the white vertex in the unitary type map $\mathfrak{m}$ connected to the half-edge labelled k. We will omit the notation $\mathfrak{m}$ if there is no ambiguity.

Notice that in a map of unitary type, the half-edges connected to black vertices are not labelled. We now explain how to label them. Consider, in a map of unitary type, an unlabelled half-edge which we denote by h. This half-edge has a face f to its left (see Figure 3.1). Starting from h, we turn around the face in the clockwise direction until we encounter a labeled half-edge connected to a white vertex, which is labelled by i. We assign to h the label i. See Figures 3.3 and 3.4.

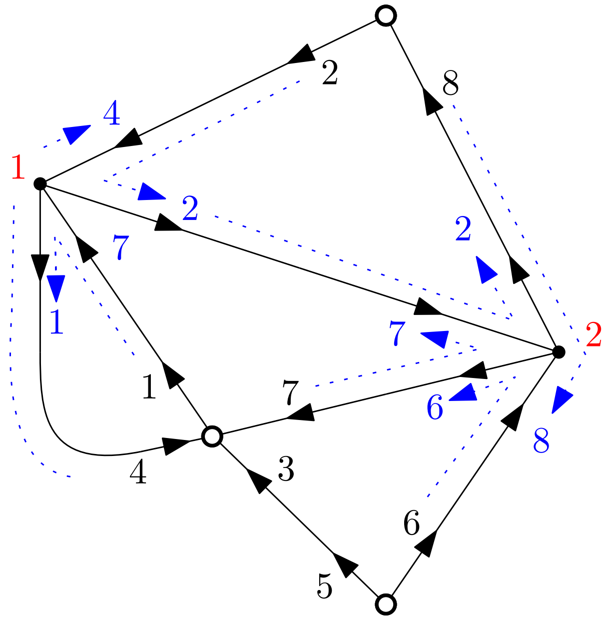


Figure 3.4: Procedure to assign labels to half-edges
The newly labelled half-edges are in blue (and follow the dotted arrows).

Notice that by Remark 3.3.10, all faces are incident to at least one white vertex, so all unlabelled half-edges can be labelled by this procedure, in a unique way.

The following Lemma will be used to prove Lemma 3.3.18.

Lemma 3.3.12. *Let h be a half-edge labelled by i. There exists a unique white half-edge h' labelled by i. If h is ingoing then h' is ingoing. If h is outgoing then h' is outgoing.*

Proof. Consider an ingoing half-edge h. The existence and uniqueness of h' is a consequence of the definition. If h is a white half-edge, the statement is obvious. If not, then consider the face f to its left. Starting from h we turn around f in the clockwise direction until we reach a white vertex w. All the vertices we encounter before w are black. The black vertices are alternated so all the half-edges such that f is at their left are ingoing as well, and so is the white half-edge h' that we reach, whose label is the same as the label of h. We proceed similarly for outgoing half-edges. □

The labels for the edges allow us to define the notion of value of a black vertex.

Definition 3.3.13. *Consider a black vertex b. Let i and j be the labels of the two outgoing half-edges at b. The **value** of the black vertex b is $\mathrm{val}(b) = \max(i, j)$.*

Definition 3.3.14. *A map of unitary type with r black vertices $b_1, \ldots, b_r$ numbered respectively $1, \ldots, r$ is **nondecreasing** if*

$$\mathrm{val}(b_1) \leq \mathrm{val}(b_2) \leq \cdots \leq \mathrm{val}(b_r).$$

Example 3.3.15. Figure 3.4 displays an example. The labels of the black vertices are in red. The values of the black vertices s_1 and s_2 are $\mathrm{val}(s_1) = 2, \mathrm{val}(s_2) = 6$.

3.3.3 Permutational model

Similarly as in Sections 2.4.1 and 3.3.1, we define a permutational model for the maps of unitary type.

Definition 3.3.16. *Let $I \subset \mathbf{N}^*$ be finite and $r \in \mathbf{N}$. Let $\mathfrak{m}$ be a map of unitary type with labels in $I \neq \emptyset$ and r black vertices.*

We define $\boldsymbol{\epsilon}_{\mathfrak{m}} = (\epsilon(i), i \in I)$ *as follows. If the white half-edge labelled* $i \in I$ *is outgoing, we set* $\epsilon(i) = +1$*, else we set* $\epsilon(i) = -1$*.*

We define $\gamma_{\mathfrak{m}}, \pi_{\mathfrak{m}}, \phi_{\mathfrak{m}} \in \mathfrak{S}(I)$ *and* $\tau_{\mathfrak{m}} = (\tau_1, \ldots, \tau_r) \in \mathfrak{S}(\epsilon_{\mathfrak{m}}^{-1}(+1))^r$ *as follows.*

- *Let* $i \in I$*. The white half-edge* h_i*, labelled* i*, is connected to a white vertex* w_i*. Starting from* h_i*, we turn in the clockwise direction around* w_i*. Let* j *be the label of the next half-edge connected to* w_i*. We set* $\gamma_{\mathfrak{m}}(i) = j$*.*

- *Let* $i \in I$*. The white half-edge* h_i *labelled* i *is connected to another half-edge* h_j*, which is labelled by* j*. We set* $\pi_{\mathfrak{m}}(i) = j$*.*

- *Let* $i \in I$*. The white half-edge labelled* i *has a face* f_i *to its left. Starting from the half-edge* i*, we turn in the counterclockwise direction around the face* f_i*. The next white half-edge with* f_i *on its left we encounter is labelled* j*. We set* $\phi_{\mathfrak{m}}(i) = j$*.*

- *Let* b_l *be the black vertex numbered* l*. The outgoing half-edges that are connected to it are labelled by* i *and* j*. We set* $\tau_l = (i\, j)$*.*

The permutations $\gamma_{\mathfrak{m}}, \pi_{\mathfrak{m}}, \phi_{\mathfrak{m}}$ are the counterparts for maps of unitary type of the permutations $\sigma_{\mathfrak{m}}, \alpha_{\mathfrak{m}}, \varphi_{\mathfrak{m}}$ defined in Construction 2.4.10.

Example 3.3.17. For the map in Figure 3.4, we have $r = 2$ and

$$\begin{aligned}
\gamma_{\mathfrak{m}} &= (1\,7\,3\,4)(5\,6)(2\,8),\\
\epsilon_{\mathfrak{m}} &= (+1, +1, -1, -1, +1, +1, -1, -1),\\
\tau_1 &= (1\,2),\ \ \tau_2 = (2\,6),\\
\pi_{\mathfrak{m}} &= (1\,7\,6\,8\,2\,4)(3\,5),\\
\phi_{\mathfrak{m}} &= (1)(2)(3\,6)(4\,8\,5)(7).
\end{aligned}$$

Lemma 3.3.18. *The permutation* $\pi_{\mathfrak{m}}$ *belongs to* $\mathfrak{S}^{(\epsilon_{\mathfrak{m}})}(I)$*, defined in Definition 3.2.7.*

Proof. An edge consists of an outgoing half-edge h attached to an ingoing half-edge h'. Assume that h is white. Let i be the label of h and j be the label of h'. We have $\pi(i) = j$. By Lemma 3.3.12, j is the label of a white ingoing half-edge. Thus, $\epsilon(i) = -1$ and $\epsilon(j) = +1$. We proceed similarly if h' is white. □

We have the following counterpart of Lemma 2.4.11.

Lemma 3.3.19. *For a unitary type map* $\mathfrak{m}$*, we have* $\gamma_{\mathfrak{m}}\pi_{\mathfrak{m}}^{-1} = \phi_{\mathfrak{m}}$*.*

Proof. Let $i \in I$ be the label of a white outgoing half-edge, and f the face at the left of the half-edge. Starting from the half-edge labelled i, we follow the boundary of the face until we encounter a white vertex. The last half-edge we traversed, which was ingoing, was labelled by j. This half-edge is connected to a outgoing half-edge labelled i. By definition, we thus have $\pi_{\mathfrak{m}}(j) = i$. The next labelled half-edge when going around f in the counterclockwise order is the half-edge following the half-edge j when turning in the clockwise direction around the white vertex. This next half-edge is thus labelled $\gamma_{\mathfrak{m}}(j) = \gamma_{\mathfrak{m}}\pi_{\mathfrak{m}}^{-1}(i)$, see Figure 3.5.

The proof is identical if i if the label of an ingoing half-edge. □

Proposition 3.3.20. *Let* I *be a finite subset of* $\mathbf{N}^*$*,* $r \in \mathbf{N}^*$*,* $\gamma \in \mathfrak{S}(I)$*, and* $\epsilon \in \{\pm 1\}^I$*. Let* $\mathfrak{m}$ *be a unitary type map with set of labels* I *and with* r *black vertices such that* $\gamma_{\mathfrak{m}} = \gamma$ *and* $\epsilon_{\mathfrak{m}} = \epsilon$*, and let* $\tau_{\mathfrak{m}} = (\tau_1, \ldots, \tau_r)$*. Then,* $\tau_r \cdots \tau_1 = \pi_{\mathfrak{m}}^{(\epsilon)}$*.*

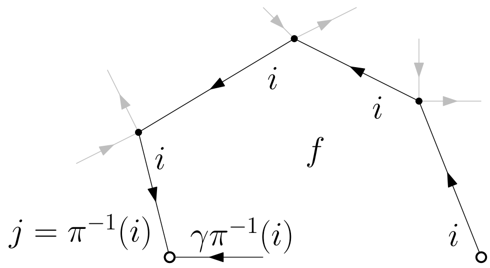


Figure 3.5: Proof of Lemma 3.3.19.

Proof. Let $k \in I$ be the label of a white outgoing half-edge connected to a vertex $w_k = u_0$. Let f be the face at its right. We construct a path starting from the half-edge labelled k as follows, see also Figure 3.6. Consider the edge $e_1 = (u_0, u_1)$ of which the half-edge labelled k is part. If u_1 is white then for all $1 \le j \le r, \tau_j(k) = k = \pi_{\mathfrak{m}}^{(\epsilon)}(k)$.

If u_1 is black, we can find vertices $u_2, u_3, \ldots, u_{p+1}$ such that $u_2, \ldots, u_p$ are black and u_{p+1} is white, and (u_j, u_{j+1}) follows (u_{j-1}, u_j) when going around the vertex u_j in the counterclockwise order. Notice that these edges are all part of the boundary of f.

Let $n_1, n_2, \ldots, n_p$ be the labels of the black vertices $u_1, \ldots, u_p$, and $k_j, 1 \le j \le p+1$ be the labels of the outgoing half-edges edges (connected to u_{j-1}) in (u_{j-1}, u_j). Notice that $1 \le n_1, \ldots, n_p \le r$ as black vertices have labels in $[r]$. By construction, we have $\tau_{n_j}(k_{j-1}) = k_j$.

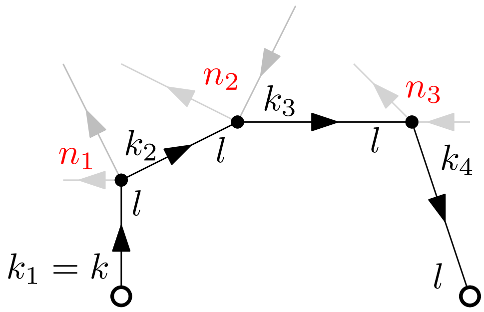


Figure 3.6: Chain of edges around the face f.

We have $\tau_{n_p}\tau_{n_{p-1}} \cdots \tau_{n_1}(k) = k_p$, so the labels of the black ingoing vertices incident to f are all equal to $l = \pi_{\mathfrak{m}}(k)$, by construction of $\pi_{\mathfrak{m}}$. We have $\pi_{\mathfrak{m}}(k) = l = \pi_{\mathfrak{m}}^{-1}(k_p)$. Thus $\tau_{n_p}\tau_{n_{p-1}} \cdots \tau_{n_1}(k) = \pi^{(\epsilon)}(k)$.

Assume now that $\tau_r \cdots \tau_1(k) \neq \tau_{n_p}\tau_{n_{p-1}} \cdots \tau_{n_1}(k)$. Let j be the minimal index such that there exists p' satisfying $n_{p'} \le j < n_{p'+1}$ (with the convention $n_{p+1} = r+1$) and $\tau_j \cdots \tau_1(k) \neq \tau_{n_{p'}} \cdots \tau_{n_1}(k)$. The index j is minimal so $j > n_{p'}$ (else we would have a contradiction as $\tau_{j-1} \cdots \tau_1(k) = \tau_{n_{p'-1}} \cdots \tau_{n_1}(k)$). We have $k_{p'} = \tau_{j-1} \cdots \tau_1(k) = \tau_{n_{p'}} \cdots \tau_{n_1}(k)$. By construction, all the half-edges labelled by $k_{p'}$ are on the boundary of a same face f', and they follow each other. We have just seen that there is such an half-edge in the edge between $u_{p'}$ and $u_{p'+1}$. The fact that $\tau_j(k_{p'}) \neq k_{p'}$ implies that there is an half-edge labelled $k_{p'}$ that is connected to the j-th black vertex. However, this edge must be before (when going around the face f') or after the edge $(u_{p'}, u_{p'+1})$ in the boundary of f'. This contradicts the fact that if there is an edge going from a black vertex i to a black vertex labelled j we have $i < j$, as $n_{p'} < j < n_{p'+1}$. □

Definition 3.3.21. *We denote by $\mathfrak{C}^r(I, \epsilon, \gamma)$ the set of nondecreasing unitary type maps $\mathfrak{m}$ with set of labels I and with r black vertices such that $\gamma_{\mathfrak{m}} = \gamma$ and $\epsilon_{\mathfrak{m}} = \epsilon$.*

Similarly, we denote by $\mathfrak{C}(g, I, \epsilon, \gamma)$ the set of nondecreasing unitary type maps $\mathfrak{m}$ with set of labels I and with genus g such that $\gamma_{\mathfrak{m}} = \gamma$ and $\epsilon_{\mathfrak{m}} = \epsilon$.

Theorem 3.3.22. *Let I be a finite subset of the positive integers, $r \in \mathbf{N}^*$, $\epsilon \in \{\pm 1\}^I$ and $\gamma \in \mathfrak{S}(I)$.*

The mapping

$$\mathfrak{C}^r(I,\epsilon,\gamma) \to \bigcup_{\pi\in\mathfrak{S}^{(\epsilon)}(I)} \{\pi\}\times \overrightarrow{\mathcal{W}}^r(\mathrm{Id},\pi^{(\epsilon)})$$
$$\mathfrak{m} \mapsto (\pi_{\mathfrak{m}}, \tau_{\mathfrak{m}})$$

is a bijection.

Proof. Lemma 3.3.18 and Proposition 3.3.20 show that this map has values in $\bigcup_{\pi\in\mathfrak{S}^{(\epsilon)}(I)}\{\pi\}\times\overrightarrow{\mathcal{W}}^r(\mathrm{Id},\pi^{(\epsilon)})$.

We now construct an inverse mapping. To do so, we explicitly construct a map corresponding to permutations π and $\boldsymbol{\tau}=(\tau_1,\dots,\tau_r)$. By Theorem 2.4.12, it suffices to construct from π and $\boldsymbol{\tau}$ the incidence relation of the underlying graph.

To this end, we introduce the set whose elements represent the half-edges $\tilde{I}=\{h_i : i\in I\}\cup\bigcup_{j=1}^r\{h_{j,1},h_{j,2},h_{j,3},h_{j,4}\}$. We can split this set into the set of ingoing and outgoing edges $\tilde{I}=\tilde{I}_{\text{in}}\cup\tilde{I}_{\text{out}}$. We have $\tilde{I}_{\text{out}}=\{h_i : i\in I, \epsilon(i)=+1\}\cup\bigcup_{j=1}^r\{h_{j,2},h_{j,4}\}$. The elements $h_{j,k}$ represent the half-edges of the black vertices of the map we are going to construct, and the elements h_i represent the half-edges of the white vertices. We are going to define a labelling function $L\colon \tilde{I}\to I$. We set for all $i\in I$, $L(h_i)=i$. The function L is constructed by induction. At the initial step, it is only defined for the white half-edges. We then define it for the black half-edges of the black vertex i at step i.

To construct a map, we use Theorem 2.4.12. We define two permutations $\sigma,\alpha\in\mathfrak{S}(\tilde{I})$ as follows. We define $\tilde{\gamma}\in\mathfrak{S}(\tilde{I})$ by $\tilde{\gamma}(h_i)=h_{\gamma(i)}$ and the identity otherwise. We set

$$\sigma=\tilde{\gamma}(h_{1,1}\,h_{1,2}\,h_{1,3}\,h_{1,4})\cdots(h_{r,1}\,h_{r,2}\,h_{r,3}\,h_{r,4})\,.$$

The permutation α is given by the following algorithm. Let $\pi\in\mathfrak{S}^{(\epsilon)}(I)$, and $\tau=(\tau_1,\dots,\tau_r)\in\overrightarrow{\mathcal{W}}^r(\mathrm{Id},\pi^{(\epsilon)})$. We consider first the permutation $\tau_1=(i_1,j_1)$, with $i_1<j_1$. We set $\alpha_1=(h_{i_1}\,h_{1,1})(h_{j_1},h_{1,3})$. We set $L(h_{1,2})=j_1$ and $L(h_{1,4})=i_1$. In terms of maps, this procedure corresponds to connecting two edges to a same black vertex, see Figure 3.7.

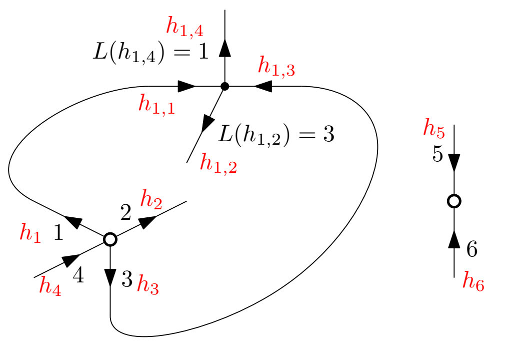


Figure 3.7: First step of the construction of the permutation α from one transposition $\tau_1=(1\,3)$, represented as a map.
The name of the half-edges are in red ($h_1,\dots,h_6$ near the white vertices and $h_{1,1},\dots,h_{1,4}$ near the black vertices), and the labels are in black. Here, $\alpha_1=(h_1\,h_{1,1})(h_3\,h_{1,3})$.

We proceed similarly to construct the black vertices labelled $2,3,\dots,r$ from the transpositions $\tau_2,\dots,\tau_r$. At the k-th step, we consider the transposition $\tau_k=(i_k\,j_k)$, with $i_k<j_k$. There is only one half-edge h (respectively h') in $\tilde{I}_{\text{out}}$ such that $L(h)=i_k$ and $\alpha_{k-1}(h)=h$ (respectively $L(h')=j_k$ and $\alpha_{k-1}(h')=h'$). We set $\alpha_k=\alpha_{k-1}(h\,h_{k,1})(h'\,h_{k,3})$.

Finally, we connect each remaining outgoing half-edge labelled i to the ingoing half-edge $\pi^{-1}(i)$. For all $i\in I$, there is a unique h such that $\alpha_r(h)=h$ and $L(h)=i$. We set $\alpha_{r+1,i}=(h\,h_{\pi^{-1}(i)})$ and define $\alpha=\alpha_r\prod_{i\in I}\alpha_{r+1,i}$. We define $\tilde{\epsilon}\in\{\pm1\}^{\tilde{I}}$ by $\tilde{\epsilon}(i)=+1$ if $i\in\tilde{I}_{\text{out}}$ and $\tilde{\epsilon}(i)=-1$ otherwise.

Theorem 3.3.5 implies that given σ,α and $\tilde{\epsilon}$, we construct a unique map $\tilde{\mathfrak{m}}$. By construction, the resulting map is of unitary type : the vertices attached to the half-edges h_i are the white vertices and the other are the black vertices. The black vertex attached to the half-edges $h_{j,k}$ is numbered j. The map is constructed such that $\pi_{\mathfrak{m}} = \pi_{\tilde{\mathfrak{m}}}$ and $\tau_{\mathfrak{m}} = \tau_{\tilde{\mathfrak{m}}}$.

Furthermore, the map is nondecreasing (recall Definition 3.3.14) as the tuple $\boldsymbol{\tau}$ is a monotone walk.

We have constructed a right inverse, so the map $\mathfrak{m} \mapsto (\pi_{\mathfrak{m}}, \tau_{\mathfrak{m}})$ is surjective. We now show that this map is injective. We show that the incidence relation of a map of unitary type $\mathfrak{m}$ is determined by the permutations. Indeed, consider a map of unitary type described by π and $\tau = (\tau_1, \dots, \tau_r)$, and an outgoing half-edge h_i labelled i. There are four cases.

- If h_i is a white half-edge such that for all j we have $\tau_j(i) = i$, then h_i is necessarily attached to the white half-edge labelled $\pi^{-1}(i)$.
- If h_i is a white half-edge and there exists k, such that $\tau_k(i) \neq i$, then h_i is necessarily connected to the k'-th black vertex, where $k' = \min\{k \colon \tau_k(i) \neq i\}$.
- If h_i is a half-edge connected to the k-th black vertex and for all $l > k$ $\tau_l(i) = i$, then h_i is necessarily attached to a white half-edge labelled $\pi^{-1}(i)$
- If h_i is a half-edge connected to the k-th black vertex and l is the smallest integer such that $l > k$ and $\tau_l(i) \neq i$, then h_i is necessarily attached to the l-black vertex.

Thus two maps of unitary type in $\mathfrak{C}^r(I, \epsilon, \gamma)$ described by the same permutations π and τ have necessarily the same edges, i.e. are identical. □

For a tuple of permutations, $(\sigma_1, \dots, \sigma_k) \in \mathfrak{S}(I)^k$, we denote the subgroup of $\mathfrak{S}(I)$ they generate by

$$\langle \sigma_1, \dots, \sigma_k \rangle \ .$$

We can associate to the triplet $(\gamma_{\mathfrak{m}}, \pi_{\mathfrak{m}}, \tau_{\mathfrak{m}})$ the group

$$G(\mathfrak{m}) = \langle \gamma_{\mathfrak{m}}, \pi_{\mathfrak{m}}, \tau_1, \dots, \tau_r \rangle, \tag{3.11}$$

where $\tau_{\mathfrak{m}} = (\tau_1, \dots, \tau_r)$.

Proposition 3.3.23. *A unitary type map $\mathfrak{m}$ with set of labels I is connected if and only if the group $G(\mathfrak{m})$ defined by (3.11) acts transitively on I.*

Proof. First, assume that $\mathfrak{m}$ is connected. Let $i, j \in I$. There is a path ρ (made up by vertices and edges) connecting the white vertices w_i and w_j. First, let us assume that ρ contains only black vertices, except for its boundary which is made up of w_i and w_j. The path encounters the black vertices $n_1, \dots, n_p$, the labels on the left of the edges that constitute ρ are $k_1, \dots, k_{p+1}$. The first and last edges are connected to w_i and w_j so $k_1 = \gamma_{\mathfrak{m}}^{m_1}(i)$ and $k_{p+1} = \gamma_{\mathfrak{m}}^{m_2}\pi_{\mathfrak{m}}^{m_3}(j)$ for some integers m_1, m_2, m_3.

Let $1 \leq i \leq p$. If $k_i = k_{i+1}$, we set $\sigma_i = \mathrm{Id}$, and if $\tau_{n_i}(k_i) = k_{i+1}$, we set $\sigma_i = \tau_{n_i}$, see Figure 3.8. Those are the only two possibilities as the half-edges connected to a black vertex labeled k, with $\tau_k = (u\, v)$ can only be labeled by u or v.

Thus, we have proved that there is $\sigma_\rho = \pi_{\mathfrak{m}}^{-m_3}\gamma_{\mathfrak{m}}^{-m_2}\sigma_p \cdots \sigma_1 \gamma_{\mathfrak{m}}^{-m_1} \in G(\mathfrak{m})$, such that $\sigma_\rho(i) = j$.

In general, any path connecting w_i and w_j can be written as the concatenation of paths with only black vertices in their interiors, we can thus construct by composition a permutation in $G(\mathfrak{m})$ that sends i to j. Thus $G(\mathfrak{m})$ is transitive.

Conversely, if $G(\mathfrak{m})$ is transitive, for any $k, l \in I$, there exists $\sigma \in G(\mathfrak{m})$ such that $\sigma(k) = l$. We can write $\sigma = \sigma_p \cdots \sigma_1$, with for all i, σ_i is one of $\gamma_{\mathfrak{m}}, \pi_{\mathfrak{m}}^{-1}, \tau_1, \dots, \tau_r$. We use this to construct a path connecting v_k to v_l. For all i, we attach to σ_i a path ρ_i starting from a half-edge labelled k_i. We set $k_1 = k$, and we will show that $k_{p+1} = l$.

- If $\sigma_i = \gamma_{\mathfrak{m}}$, ρ_i is the empty path, and $k_{i+1} = \gamma_{\mathfrak{m}}(k_i)$.

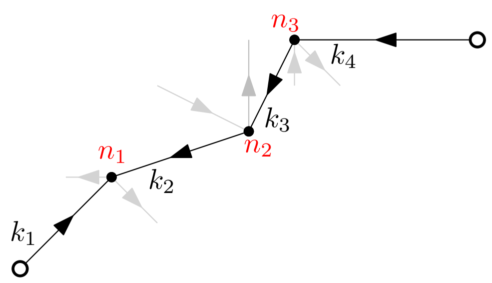


Figure 3.8: Three situations for σ_i.
We set $\sigma_1 = \tau_{n_1}$, $\sigma_2 = \mathrm{Id}$, and $\sigma_3 = \tau_{n_3}$. The black vertices are labelled n_1, n_2, n_3 in red. In grey are the two half-edges that do not play a role for each black vertex.

- If $\sigma_i = \pi_{\mathfrak{m}}^{-1}$, ρ_i is the path connecting the half-edge k_i to the half-edge $\pi^{-1}(k_i)$. Such a path exists by the propagation of labels procedure. We set $k_{i+1} = \pi^{-1}(k_i)$.
- If $\sigma_i = \tau_{n_i}$, for some n_i, and $\tau_{n_i}(k_i) = k_i$, then ρ_i is the empty path and $k_{i+1} = k_i$.
- If $\sigma_i = \tau_{n_i}$, for some n_i, and $\tau_{n_i}(k_i) \neq k_i$, then we set $k_{i+1} = \tau_{n_i}(k_i)$. Both k_i and k_{i+1} are labels of outgoing half-edges. We set ρ_i to be the path that starts from the half-edge k_i, follows the half-edges labelled k_i until it reaches the black vertex n_i, and then follows the half-edges labelled k_{i+1} until the half-edge k_{i+1}, and the vertex $w_{k_{i+1}}$.

We have constructed a path going from the half-edge i to the half-edge $k_{p+1} = \sigma(i) = j$, as wanted. □

3.3.4 Expression of the moments in terms of maps of unitary type

Theorem 3.3.22 allows us to rewrite the expression for the moments given in Corollary 3.2.13 (see Definition 3.2.5 for relevant notation).

Corollary 3.3.24. *Let $N \geq 1$ be an integer, $\boldsymbol{P} = (P_1, \dots, P_l) \in \mathcal{X}^l$ be monomials with $m = \frac{1}{2} \deg \boldsymbol{P} \leq N$. The moments under the Haar measure μ_0^N (see Definition 3.2.1) admit the following expansion*

$$\alpha_{0,l}^N(P_1, \dots, P_l) = \sum_{r \geq 0} \frac{(-1)^r}{N^{r+m}} \sum_{\mathfrak{m} \in \mathfrak{C}^r([2m], \boldsymbol{\epsilon_P}, \gamma_{\boldsymbol{P}})} \mathrm{Tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M_P}).$$

Furthermore, the series is absolutely convergent.

The weights $\mathrm{Tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M_P})$ can be interpreted as product of weights given by the faces of the map $\mathfrak{m}$, see Figure 3.9.

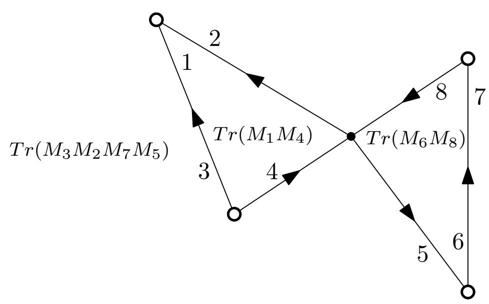


Figure 3.9: A map with its weights
This weighted map gives (up to a sign) a contribution from the sum $\alpha_{0,4}^{(0),N}(M_1u^{-1}M_2u^{-1}, M_3uM_4u, M_5u^{-1}M_6u, M_7u^{-1}M_8u)$.

Proof of Corollary 3.3.24. Recall the expression of Corollary 3.2.13:

$$\alpha_{0,l}^N(P_1,\dots,P_l) = \sum_{r\geq 0} \frac{(-1)^r}{N^{r+m}} \sum_{\pi\in\mathfrak{S}_{2m}^{(\epsilon_{\boldsymbol{P}})}} \mathrm{Tr}_{\gamma_{\boldsymbol{P}}\pi^{-1}}(\boldsymbol{M}_{\boldsymbol{P}}) \overrightarrow{w}^r(\mathrm{Id},\pi^{(\epsilon_{\boldsymbol{P}})}).$$

By definition of $\overrightarrow{w}^r(\mathrm{Id},\pi^{(\epsilon_{\boldsymbol{P}})})$, we can rewrite this as

$$\begin{aligned}\alpha_{0,l}^N(P_1,\dots,P_l) &= \sum_{r\geq 0} \frac{(-1)^r}{N^{r+m}} \sum_{\substack{\pi\in\mathfrak{S}_{2m}^{(\epsilon_{\boldsymbol{P}})}\\(\tau_1,\dots,\tau_r)\in\overrightarrow{\mathcal{W}}^r(\mathrm{Id},\pi^{(\epsilon_{\boldsymbol{P}})})}} \mathrm{Tr}_{\gamma_{\boldsymbol{P}}\pi^{-1}}(\boldsymbol{M}_{\boldsymbol{P}})\\ &= \sum_{r\geq 0} \frac{(-1)^r}{N^{r+m}} \sum_{\mathfrak{m}\in\mathfrak{C}^r([2m],\boldsymbol{\epsilon}_{\boldsymbol{P}},\gamma_{\boldsymbol{P}})} \mathrm{Tr}_{\gamma_{\boldsymbol{P}}\pi_{\mathfrak{m}}^{-1}}(\boldsymbol{M}_{\boldsymbol{P}}),\end{aligned}$$

where we used Theorem 3.3.22 in the last line.

We get the result by using Lemma 3.3.19, which gives $\gamma_{\boldsymbol{P}}\pi_{\mathfrak{m}}^{-1} = \phi_{\mathfrak{m}}$. □

Definition 2.2.5 and Corollary 3.3.24 allow us to express the cumulants in terms of maps of unitary types. We deduce the following Lemma.

Lemma 3.3.25. *Let $N \geq 1$ be an integer, $\boldsymbol{P} = (P_1,\dots,P_l) \in \mathcal{X}^l$ be monomials with $m = \frac{1}{2}\deg \boldsymbol{P}$. The cumulants admit the expansion*

$$\mathcal{W}_{0,l}^N(P_1,\dots,P_l) = \sum_{r\geq 0} \frac{(-1)^r}{N^{r+m}} \sum_{\substack{\mathfrak{m}\in\mathfrak{C}^r([2m],\boldsymbol{\epsilon}_{\boldsymbol{P}},\gamma_{\boldsymbol{P}})\\ \mathfrak{m}\text{ is connected}}} \mathrm{Tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M}_{\boldsymbol{P}}).$$

Furthermore, the series is absolutely convergent.

Proof. We show the formula by induction using Corollary 3.3.24. Notice first that when $l = 1$, $\alpha_{0,1}^N(P_1) = \mathcal{W}_{0,1}^N(P_1)$ and the maps in $\mathfrak{C}^r([2m],\boldsymbol{\epsilon}_{\boldsymbol{P}},\gamma_{\boldsymbol{P}})$ are connected.

Then, we notice that a map can be decomposed into its connected components. This decomposition gives a partition of the set of labels of half-edges. Each block contains the labels appearing in one connected component. Using Definitions 2.2.5 and Definition 3.2.1, we obtain that

$$\begin{aligned}\mathcal{W}_{0,l}^N(P_1,\dots,P_l) &= \alpha_{0,l}^N(P_1,\dots,P_l) - \sum_{\substack{\Pi\in\mathcal{P}([l])\\|\Pi|\geq 2}} \prod_{B\in\Pi} \mathcal{W}_{0,|B|}^N(P_i, i\in B)\\ &= \sum_{r\geq 0} \frac{(-1)^r}{N^{r+m}} \sum_{\mathfrak{m}\in\mathfrak{C}^r([2m],\boldsymbol{\epsilon}_{\boldsymbol{P}},\gamma_{\boldsymbol{P}})} \mathrm{Tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M}_{\boldsymbol{P}})\\ &\quad - \sum_{r\geq 0} \frac{(-1)^r}{N^{r+m}} \sum_{\substack{\mathfrak{m}\in\mathfrak{C}^r([2m],\boldsymbol{\epsilon}_{\boldsymbol{P}},\gamma_{\boldsymbol{P}})\\ \mathfrak{m}\text{ has at least 2 connected components}}} \mathrm{Tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M}_{\boldsymbol{P}}).\end{aligned}$$

Hence the result. □

Remark 3.3.26. The formulae imply that we can express moments and cumulants with respect to the Haar measure as a weighted sum over maps. The maps are the nondecreasing maps of unitary type whose local structure (i.e. how the half-edges are attached to the vertices, but not how the half-edges are attached together) is determined by $\gamma_{\boldsymbol{P}}^{-1}$ and $\boldsymbol{\epsilon}_{\boldsymbol{P}}$. To each face is associated a weight, which is the trace of a certain word in the matrices of $\boldsymbol{M}_{\boldsymbol{P}}$, times a sign.

A topological expansion for the Haar measure. We now rewrite Lemma 3.3.25 as a sum over the genus g of the maps rather than on the number of black vertices r. We will see that this gives us an expansion in powers of $\frac{1}{N^2}$. We first recall Euler's formula

$$2 - 2g(\mathfrak{m}) = V(\mathfrak{m}) - E(\mathfrak{m}) + F(\mathfrak{m}), \tag{3.12}$$

where $V(\mathfrak{m})$, $E(\mathfrak{m})$ and $F(\mathfrak{m})$ are the number of vertices, edges and faces of a map $\mathfrak{m}$, and $g(\mathfrak{m})$ is its genus. In the case of a map of unitary type labelled by a set of $2m$ integers, and with r black vertices, we have

- $c(\gamma_{\mathfrak{m}})$ white and r black vertices,
- $2m$ white half-edges and $4r$ half-edges out of black vertices, for a total of $m + 2r$ edges,
- $c(\phi_{\mathfrak{m}})$ faces (see Definition 2.4.1).

Thus, we get

$$2 - 2g(\mathfrak{m}) = (c(\gamma_{\mathfrak{m}}) + r) - (m + 2r) + c(\phi_{\mathfrak{m}}) = c(\gamma_{\mathfrak{m}}) + c(\phi_{\mathfrak{m}}) - m - r. \tag{3.13}$$

A change of variable in the sum of Lemma 3.3.25, and the identities $l = c(\gamma_{\boldsymbol{P}})$ and $\mathrm{Tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M_P}) = N^{c(\phi_{\mathfrak{m}})}\,\mathrm{tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M_P})$ give the following Proposition.

Proposition 3.3.27. *Let $N \geq 1$ be an integer, $\boldsymbol{P} = (P_1, \ldots, P_l) \in \mathcal{X}^l$ be monomials with $m = \frac{1}{2}\deg \boldsymbol{P}$. The cumulants admit the expansion*

$$\mathcal{W}_{0,l}^{N}(P_1, \ldots, P_l) = N^{2-l}(-1)^{m+l} \sum_{g \geq 0} \frac{1}{N^{2g}} \sum_{\substack{\mathfrak{m} \in \mathfrak{C}(g,[2m],\boldsymbol{\epsilon_P},\gamma_{\boldsymbol{P}}) \\ \mathfrak{m} \text{ is connected}}} (-1)^{c(\phi_{\mathfrak{m}})}\,\mathrm{tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M_P}).$$

Furthermore, the series is absolutely convergent.

Notice that this expansion is in terms of the normalized trace $\mathrm{tr} = \frac{1}{N}\mathrm{Tr}$. The factors with the trace are bounded by 1 if we assume $\|A_i^N\| \leq 1$ for all $1 \leq i \leq N$ and $N \geq 1$.

Remark 3.3.28. The sum in Proposition 3.3.27 is in general not finite. Indeed, even for $l = 1$ and $P_1 = AUBU^*$, the sum contains terms of arbitrary genus. They appear for instance because of the factorization of the identity $\mathrm{Id} = (1\,2)^{2k}$, for all $k \geq 0$.

Definition 3.3.29. *Let $N \geq 1$ be an integer, $\boldsymbol{P} = (P_1, \ldots, P_l) \in \mathcal{X}^l$ be monomials with $m = \frac{1}{2}\deg \boldsymbol{P}$. The term of order $2g$ in the expansion of the cumulant is denoted by*

$$\mathcal{M}_{0,l}^{(g),N}(P_1, \ldots, P_l) = (-1)^{m+l} \sum_{\substack{\mathfrak{m} \in \mathfrak{C}(g,[2m],\boldsymbol{\epsilon_P},\gamma_{\boldsymbol{P}}) \\ \mathfrak{m} \text{ is connected}}} (-1)^{c(\phi_{\mathfrak{m}})}\,\mathrm{tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M_P}).$$

We extend this definition to all monomials in $\mathcal{A}$ by setting for $P_1, \ldots, P_l \in \mathcal{X}$ and M a word in the (a_i, a_i^*),

$$\mathcal{M}_{0,l}^{(g),N}(P_1, \ldots, P_{i-1}, P_iM, P_{i+1}, \ldots, P_l) = \mathcal{M}_{0,l}^{(g),N}(P_1, \ldots, P_{i-1}, MP_i, P_{i+1}, \ldots, P_l),$$

for all $1 \leq i \leq l$.

The last property is enforced so that $\mathcal{M}_{0,l}^{(g),N}$ has a property of cyclicity, as does the trace.

Stationary distribution of the $(A_i^N)_{1\le i\le p}$. Let us consider a particular choice for the sequence of matrices $(A_i^N)_{1\le i\le p, N\ge 1}$. Fix a family of p matrices of fixed size $M\times M$, $(A_i^M)_{1\le i\le p}$, and consider the sequence of matrices $(A_i^{qM})_{1\le i\le p, q\ge 1}$, where A_i^{qM} is the block-diagonal matrix with q blocks, whose blocks are A_i^M. When considering the sums of maps for $N=qM$, the traces $\mathrm{tr}_\phi(\boldsymbol{M})$ no longer depend on q or $N=qM$.

In the case of zero potential ($V=0$), by Proposition 3.3.27, the renormalized cumulant $\tilde{\mathcal{W}}_{0,1}^N$ converges with limit

$$\lim_{q\to\infty}\tilde{\mathcal{W}}_{0,1}^{qM}(P)=\mathcal{M}_{0,1}^{(0),M}(P),$$

for $P\in\mathcal{A}$. This fact allows us to prove the following Lemma.

Lemma 3.3.30. *Fix $N\in\mathbf{N}^*$. Assume that $\|A_i^N\|\le 1$ for all $1\le i\le p$. Let $P\in\mathcal{X}$ be a monomial. We have for all choices of $(A_i^N)_{1\le i\le p}$ that*

$$|\mathcal{M}_{0,1}^{(0),N}(P)|\le 1.$$

Proof. By the previous remark, we have for the choice of with the $(A_i^N)_{1\le i\le p}$ block diagonal as above,

$$|\mathcal{M}_{0,1}^{(0),M}(P)|=\lim_{N\to\infty}|\tilde{\mathcal{W}}_{0,1}^N(P)|=\lim_{N\to\infty}\mathbb{E}\left[\mathrm{tr}(P)\right]\le 1,$$

as $\|P\|\le 1$. □

More generally, with the $(A_i^N)_{1\le i\le p}$ block diagonal as above, we have

$$\tilde{\mathcal{W}}_{0,l}^{qM}(\boldsymbol{P})=(-1)^{m+l}\sum_{g\ge 0}\frac{1}{(qM)^{2g}}\sum_{\substack{\mathfrak{m}\in\mathfrak{C}(g,[2m],\boldsymbol{\epsilon_P},\gamma_{\boldsymbol{P}})\\ \mathfrak{m}\text{ is connected}}}(-1)^{c(\phi_\mathfrak{m})}\,\mathrm{tr}_{\phi_\mathfrak{m}}(\boldsymbol{M_P})\,,$$

where $\mathrm{tr}_{\phi_\mathfrak{m}}(\boldsymbol{M_P})$ does not depend on q. This implies the following Lemma.

Lemma 3.3.31. *For all $N\ge 1$, $g\ge 0$, $l\ge 1$, and $\boldsymbol{P}\in\mathcal{X}^l$, we have the following properties.*

(i) (Traciality) For all $Q\in\mathcal{X}$,

$$\mathcal{M}_{0,l}^{(g),N}(P_1,\dots,P_{l-1},P_lQ)=\mathcal{M}_{0,l}^{(g),N}(P_1,\dots,P_{l-1},QP_l)\,.$$

(ii) (Symmetry) For all permutation $\sigma\in\mathfrak{S}_l$,

$$\mathcal{M}_{0,l}^{(g),N}(P_1,\dots,P_l)=\mathcal{M}_{0,l}^{(g),N}(P_{\sigma(1)},\dots,P_{\sigma(l)})\,.$$

(iii) (Simplification) We have

$$\mathcal{M}_{0,l}^{(g),N}(P_1,\dots,P_{l-1},u^*P_lu)=\mathcal{M}_{0,l}^{(g),N}(P_1,\dots,P_l)\,.$$

(iv) (Conjugation) We have

$$\mathcal{M}_{0,l}^{(g),N}(P_1^*,\dots,P_l^*)=\overline{\mathcal{M}_{0,l}^{(g),N}(P_1,\dots,P_l)}\,.$$

Proof. Consider the series

$$G(\hbar)=(-1)^{m+l}\sum_{g\ge 0}\hbar^{2g}\sum_{\substack{\mathfrak{m}\in\mathfrak{C}(g,[2m],\boldsymbol{\epsilon_P},\gamma_{\boldsymbol{P}})\\ \mathfrak{m}\text{ is connected}}}(-1)^{c(\phi_\mathfrak{m})}\,\mathrm{tr}_{\phi_\mathfrak{m}}(\boldsymbol{M_P}),$$

where the polynomials in the tuple $\boldsymbol{M_P}$ are evaluated at the matrices $(A_i^M,(A_i^M)^*)_{i\in[2m]}$. Proposition 3.3.27 implies that $G(1/qM)=\tilde{\mathcal{W}}_{0,l}^N(\boldsymbol{P})$.

Thus, as the renormalized cumulant under the Haar measure $\tilde{\mathcal{W}}_{0,l}^N=N^{l-2}\mathcal{W}_{0,l}^N$ satisfies all four properties, and the set $\{1/qM\}_{q\ge 1}$ has an accumulation point, we get the result. □

3.3.5 Formal topological expansion

When the potential V is not zero, we expect to have an expansion of the free energy as in Proposition 3.3.27. Let us now consider a potential of the form $V = \sum_{i=1}^{k} z_i q_i$, with $\boldsymbol{z} = (z_1, \dots, z_k) \in \mathbb{C}^k$ and $\boldsymbol{q} = (q_1, \dots, q_k) \in \mathcal{X}^k$.

Proposition 3.3.27 motivates the introduction of the formal series

$$\begin{aligned} F_V^{N,f} &= \frac{1}{N^2} \sum_{\boldsymbol{n}\in\mathbf{N}^k} \frac{(-\boldsymbol{z})^{\boldsymbol{n}}}{\boldsymbol{n}!} \tilde{\mathcal{W}}^N_{0,\sum_i n_i}(\boldsymbol{q_n}) \\ &= \sum_{g\geq 0} \frac{1}{N^{2g}} \sum_{\boldsymbol{n}\in\mathbf{N}^k} \frac{\boldsymbol{z}^{\boldsymbol{n}}}{\boldsymbol{n}!} (-1)^{\deg \boldsymbol{q_n}} \sum_{\substack{\mathfrak{m}\in\mathfrak{C}(g,[2\deg \boldsymbol{q_n}],\boldsymbol{\epsilon_{q_n}},\gamma(\boldsymbol{q_n}))\\ \mathfrak{m}\text{ is connected}}} (-1)^{c(\phi_{\mathfrak{m}})} \operatorname{tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M_{q_n}}), \end{aligned} \tag{3.14}$$

where we use the notation $\boldsymbol{q_n} = (\underbrace{q_1, \dots, q_1}_{n_1 \text{ times}}, \dots, \underbrace{q_k, \dots, q_k}_{n_k \text{ times}})$ for $\boldsymbol{n} = (n_1, \dots, n_k)$, as well as $\boldsymbol{z}^{\boldsymbol{n}} = \prod_{i=1}^{k} z_i^{n_i}$ and $\boldsymbol{n}! = \prod_{i=1}^{k} n_i!$.

Similarly, we introduce formal series corresponding to the cumulants.

Definition 3.3.32. *Let $N \in \mathbf{N}^*$, $\boldsymbol{P} = (P_1, \dots, P_l) \in \mathcal{X}^l$ be monomials with $m = \frac{1}{2}\deg \boldsymbol{P}$. The **formal cumulant** of $\boldsymbol{P}$ is the formal series*

$$\mathcal{M}^N_{V,l}(P_1, \dots, P_l) = \sum_{g\geq 0} \frac{1}{N^{2g}} \mathcal{M}^{(g),N}_{V,l}(P_1, \dots, P_l),$$

where the g-th term is

$$\begin{aligned} &\mathcal{M}^{(g),N}_{V,l}(P_1, \dots, P_l) \\ &\quad = \sum_{\boldsymbol{n}\in\mathbf{N}^k} \frac{\boldsymbol{z}^{\boldsymbol{n}}}{\boldsymbol{n}!} (-1)^{\deg \boldsymbol{q_n} + \deg \boldsymbol{P}} \sum_{\substack{\mathfrak{m}\in\mathfrak{C}(g,[\deg \boldsymbol{q_n}+\deg \boldsymbol{P}],\boldsymbol{\epsilon_{q_nP}},\gamma_{\boldsymbol{q_nP}})\\ \mathfrak{m}\text{ is connected}}} (-1)^{c(\phi_{\mathfrak{m}})} \operatorname{tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M_{q_nP}}), \end{aligned}$$

where $\boldsymbol{q_nP}$ is the concatenation of the two tuples $\boldsymbol{q_n}$ and $\boldsymbol{P}$. In particular, $\gamma_{\boldsymbol{q_nP}}$ is the permutation defined in (3.8) associated to the tuple $\boldsymbol{q_nP}$.

Notice that the total numbers of u and u^* (or white half-edges) is $\sum_i n_i \deg q_i + 2m$, and the number of white vertices is $\sum_i n_i + l$. The case of the formal free energy corresponds to $m = 0$, $l = 0$.

At this point, it is not clear whether the series $\mathcal{M}^{(g),N}_{V,l}(P_1, \dots, P_l)$ converge. It will be shown in Section 3.5.3.

In Section 3.7, we will show that in the asymptotic regime, the cumulant $\mathcal{W}^N_{V,l}(P_1, \dots, P_l)$ coincides with the formal cumulant up to an arbitrary order, for $\boldsymbol{z}$ small enough.

3.3.6 Alternated polynomials and Hurwitz numbers

In this section, we consider a particular case, that is we assume that all polynomials are alternated monomials (see Definition 3.3.33). In particular, this covers the case of a potential of the form $V = zAU^N B(U^N)^*$ encountered in the HCIZ integral. In [GGN14], the HCIZ integral had been expressed in terms of monotone double Hurwitz numbers. In the multi-matrix case, results relating the more general tensor HCIZ integral to the Hurwitz numbers have been obtained in [CGL23].

Here we consider only the case where we have a single unitary matrix.

Definition 3.3.33. *A monomial $P \in \mathcal{A}$ is said to be **alternated** if it can be written*

$$P = B_1 u C_1 u^{-1} \cdots B_m u C_m u^{-1},$$

with $B_i, C_i, 1 \leq i \leq m$ words in $a_1, a_1^*, \dots, a_p, a_p^*$.

In this section, we will assume that all the polynomials involved $(P_1, \dots, P_l, q_1, \dots, q_k)$ are alternated monomials. We write as before $\boldsymbol{P} = (P_1, \dots, P_l)$. We now explain how we can give different expressions for the renormalized cumulants in this case. To do so we reason only using the permutational model of the maps of unitary type.

In this case, $\boldsymbol{\epsilon} = (+1, -1, +1, -1, \dots)$, and we have $\gamma_{\boldsymbol{P}}(\boldsymbol{\epsilon}^{-1}(+1)) = \boldsymbol{\epsilon}^{-1}(-1)$ and $\gamma_{\boldsymbol{P}}(\boldsymbol{\epsilon}^{-1}(-1)) = \boldsymbol{\epsilon}^{-1}(+1)$. Thus, $\gamma_{\boldsymbol{P}} \in \mathfrak{S}_{2m}^{(\boldsymbol{\epsilon})}$. We define $\tilde{\gamma} = \gamma_{\boldsymbol{P}}^2|_{\boldsymbol{\epsilon}^{-1}(+1)}$.

In particular, this implies that for all $\mathfrak{m} \in \mathfrak{C}(g, [2m], \boldsymbol{\epsilon}, \gamma_{\boldsymbol{P}})$, we have $\phi_{\mathfrak{m}}(\boldsymbol{\epsilon}^{-1}(+1)) = \boldsymbol{\epsilon}^{-1}(+1)$ and $\phi_{\mathfrak{m}}(\boldsymbol{\epsilon}^{-1}(-1)) = \boldsymbol{\epsilon}^{-1}(-1)$. That is, we can write $\phi_{\mathfrak{m}}$ as a product of two permutations, one, $\phi_{\mathfrak{m}}^+$, having its support in $\boldsymbol{\epsilon}^{-1}(+1)$, and the other, $\phi_{\mathfrak{m}}^-$, having its support in $\boldsymbol{\epsilon}^{-1}(-1)$.

Write $\tau_{\mathfrak{m}} = (\tau_1, \dots, \tau_r)$. We then notice that the group generated by $\gamma_{\boldsymbol{P}}, \phi_{\mathfrak{m}}, \tau_1, \dots \tau_r$ is transitive if and only if the group generated by $\gamma_{\boldsymbol{P}}, \phi_{\mathfrak{m}}^+, \phi_{\mathfrak{m}}^-, \tau_1, \dots \tau_r$ is transitive. As conjugating the elements of a group by one of the elements does not change the group, we have

$$\langle \gamma_{\boldsymbol{P}}, \phi_{\mathfrak{m}}^+, \phi_{\mathfrak{m}}^-, \tau_1, \dots, \tau_r \rangle = \langle \gamma_{\boldsymbol{P}}, \phi_{\mathfrak{m}}^+, \gamma_{\boldsymbol{P}}^{-1} \phi_{\mathfrak{m}}^- \gamma_{\boldsymbol{P}}, \tau_1, \dots, \tau_r \rangle .$$

Now, we remark that the subgroup of $\mathfrak{S}_{2m}$ on the right-hand side is transitive if and only if the subgroup $\langle \tilde{\gamma}, \phi_{\mathfrak{m}}^+, \gamma_{\boldsymbol{P}}^{-1} \phi_{\mathfrak{m}}^- \gamma_{\boldsymbol{P}}, \tau_1, \dots, \tau_r \rangle$ of $\mathfrak{S}(\epsilon^{-1}(+1))$ is transitive.

This remark allows us to rewrite the sum of Definition 3.3.29,

$$\begin{aligned}
\mathcal{W}_{0,l}^{(g),N}(P_1, \dots, P_l) &= (-1)^m \sum_{\substack{\mathfrak{m} \in \mathfrak{C}(g,[2m],\boldsymbol{\epsilon}_{\boldsymbol{P}} \gamma_{\boldsymbol{P}}) \\ \mathfrak{m} \text{ connected}}} (-1)^{c(\phi_{\mathfrak{m}})} \operatorname{tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M}_P) \\
&= (-1)^m \sum_{\substack{\phi^+, \phi^- \in \mathfrak{S}_m \\ \tau \in \overrightarrow{\mathcal{W}}_g(\mathrm{Id}, (\phi^+)^{-1} \tilde{\gamma} \phi^-) \\ \langle \tilde{\gamma}, \phi^+, \phi^-, \tau_1, \dots, \tau_r \rangle \text{ transitive}}} (-1)^{c(\phi^+) + c(\phi^-)} \operatorname{tr}_{\phi^+}(\boldsymbol{B}_{\boldsymbol{P}}) \operatorname{tr}_{\phi^-}(\boldsymbol{C}_{\boldsymbol{P}}),
\end{aligned}$$

where we introduced the notation

$$\begin{aligned}
\boldsymbol{B}_{\boldsymbol{P}} &= (M_i; i \in \boldsymbol{\epsilon}^{-1}(+1)) \\
\boldsymbol{C}_{\boldsymbol{P}} &= (M_i; i \in \boldsymbol{\epsilon}^{-1}(-1)),
\end{aligned}$$

if $\boldsymbol{M}_{\boldsymbol{P}} = (M_i)_{1 \le i \le \deg \boldsymbol{P}}$.

Definition 3.3.34. *Let $\rho, \gamma, \sigma \in \mathfrak{S}_m$. The r-**th monotone triple Hurwitz number** associated to ρ, γ, σ, denoted by $\overrightarrow{h}^r(\rho, \gamma, \sigma)$, is the number of r-uplet of transpositions $(\tau_1, \dots, \tau_r) \in \mathfrak{S}_m^r$ such that*

- $\tau_r \cdots \tau_1 = \rho\gamma\sigma$;
- $\mathrm{val}(\tau_1) \le \mathrm{val}(\tau_2) \le \cdots \le \mathrm{val}(\tau_r)$;
- *the group $\langle \gamma, \rho, \sigma, \tau_1, \dots, \tau_r \rangle \subset \mathfrak{S}_m$ is transitive.*

When g satisfies the Euler equation

$$2 - 2g = c(\gamma) + c(\rho) + c(\sigma) - r - m,$$

we set $\overrightarrow{h}_g(\gamma, \sigma, \rho) = \overrightarrow{h}^r(\gamma, \sigma, \rho)$.

This gives us the following Proposition.

Proposition 3.3.35. *Let $\boldsymbol{P} = (P_1, \dots, P_l)$ be alternated monomials. We have*

$$\mathcal{W}_{0,l}^{(g),N}(P_1, \dots, P_l) = (-1)^m \sum_{\sigma, \rho \in \mathfrak{S}_m} (-1)^{c(\sigma)+c(\rho)} \operatorname{tr}_\rho(\boldsymbol{B}_{\boldsymbol{P}}) \operatorname{tr}_\sigma(\boldsymbol{C}_{\boldsymbol{P}}) \overrightarrow{h}_g(\rho^{-1}, \tilde{\gamma}, \sigma). \tag{3.15}$$

Remark 3.3.36. In the case of the HCIZ integral, we have $\tilde{\gamma} = \mathrm{Id}$, thus the monotone triple Hurwitz numbers reduce to the monotone double Hurwitz numbers.

Remark 3.3.37. Notice that when all the polynomials are alternated, all the white vertices in the unitary type maps involved are alternated vertices (see Definition 3.3.6).

Remark 3.3.38. In this particular case, the maps of unitary type bear similarity with the ribbon graphs introduced in [Joh12].

3.4 Intermezzo: Voiculescu's Theorem and maps of unitary type

As in Section 3.3.6, we consider polynomials which are balanced. This is the case we have to consider to give a proof of Voiculescu's Theorem in terms of maps of unitary type. We find it interesting to show how the point of view of maps of unitary type give a short proof of Theorem 2.2.12. This connects two methods for computing asymptotically the moments of a pair of random matrices X and Y in terms of product of moments of X and of moments of Y:

- the asymptotic freeness property;
- the computation of the leading order of the moments in terms of maps of unitary type.

Beyond this question, the particular case of balanced polynomials is the natural setup to study multi-matrix Hermitian models. Indeed, given two independent, unitarily-invariant, random matrices X and Y, we have that for any $P \in \langle x, y\rangle$ that after diagonalizing X and Y:

$$\mathbb{E}\left[\operatorname{tr} P(X,Y)\right] = \mathbb{E}\left[\operatorname{tr} P(D_X, U D_Y U^*)\right],$$

where U is a Haar-distributed unitary matrix, and D_X and D_Y are the diagonal matrices of the eigenvalues of X and Y respectively. If P is a monomial, the trace term can be written as:

$$\operatorname{tr} P(D_X, U D_Y U^*) = \operatorname{tr} D_X^{k_1} U D_Y^{l_1} U^* \cdots D_X^{k_d} U D_Y^{l_d} U^*,$$

with $k_1, \dots, k_d, l_1, \dots, l_d \geq 0$ integers. The right hand side term is a balanced monomial.

We now get back to Voiculescu's Theorem, Theorem 2.2.12.

Proof of Theorem 2.2.12. Let $(X^N), (Y^N)$ be two sequences of centered deterministic matrices, with X^N and Y^N of size $N \times N$. Assume (X^N) and (Y^N) converge in distribution, in the sense of Definition 2.2.10, with limiting distribution τ_X and τ_Y. Consider a balanced monomial of degree $2m$:

$$P(X,Y,U,U^*) = \overrightarrow{\prod_{i=1}^{m}} \left(X^{k_i} - \operatorname{tr} X^{k_i}\right) U \left(Y^{l_i} - \operatorname{tr} Y^{l_i}\right) U^*,$$

with $k_1, \dots, k_m, l_1, \dots, l_m \geq 1$ integers. We write

$$\begin{aligned} \boldsymbol{X} &= \left(X^{k_1} - \operatorname{tr} X^{k_1}\right) \otimes \cdots \otimes \left(X^{k_m} - \operatorname{tr} X^{k_m}\right) \\ \boldsymbol{Y} &= \left(Y^{l_1} - \operatorname{tr} Y^{l_1}\right) \otimes \cdots \otimes \left(Y^{l_m} - \operatorname{tr} Y^{l_m}\right). \end{aligned}$$

By Proposition 3.3.27, the leading order of the expectation of $\operatorname{tr} P$ is

$$\begin{aligned} \mathbb{E}\left[\operatorname{tr}(P)\right] &= \mathcal{W}_{0,1}^{(0),N}(P) + \mathcal{O}\big(N^{-2}\big) \\ &= (-1)^{m+1} \sum_{\substack{\mathfrak{m} \in \mathfrak{C}(0,[2m],\boldsymbol{\epsilon_P},\gamma_{\boldsymbol{P}}) \\ \mathfrak{m} \text{ is connected}}} (-1)^{c(\phi_{\mathfrak{m}})} \operatorname{tr}_{\phi_{\mathfrak{m}}^+}(\boldsymbol{X}) \operatorname{tr}_{\phi_{\mathfrak{m}}^-}(\boldsymbol{Y}) + \mathcal{O}\big(N^{-2}\big). \end{aligned} \tag{3.16}$$

Let $\mathfrak{m} \in \mathfrak{C}(0,[2m],\boldsymbol{\epsilon_P},\gamma_{\boldsymbol{P}})$ be a connected map of unitary type. We claim that one of the face of $\mathfrak{m}$ in incident to exactly one white half-edge. Assuming this, we see that in this case

$$\operatorname{tr}_{\phi_{\mathfrak{m}}^+}(\boldsymbol{X}) \operatorname{tr}_{\phi_{\mathfrak{m}}^-}(\boldsymbol{Y}) = 0$$

as the product of traces contains either a factor $\operatorname{tr}\big(X^{k_i} - \operatorname{tr} X^{k_i}\big) = 0$ or $\operatorname{tr}\big(Y^{l_i} - \operatorname{tr} Y^{l_i}\big)$. Thus, all the terms in the leading order in (3.16) are 0, and we have shown the asymptotic freeness result.

We now show the claim. Denote by h_i the half-edge labelled by $i \in [2m]$. Define

$$d = \begin{cases} \max\left\{k \in [2m] \colon \pi_{\mathfrak{m}}(k) = k-1\right\} & \text{if } \left\{k \in [2m] \colon \pi_{\mathfrak{m}}(k) = k-1\right\} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Consider the sequence $(\pi(k))_{k\in[2m]}$. We are going to show that

$$\pi(m) < \pi(m-1) < \cdots < \pi(d) < d < d+1 < \cdots < 2m.$$

Fix $i \in \{d+1, \dots, 2m\}$ such that $\pi(i) < i$, and let f_i be the face at the left of h_i. By definition, when going around f in the counterclockwise direction, $h_{\pi(i)}$ is the first white half-edge after h_i. By planarity, the part of boundary of f_i between these two half-edges separates the map into two simply connected domains. This implies that for all $\pi(i) < j < i$, we have that $\pi(i) < \pi(j) < i$. Considering $i = 2m$ and $j = 2m-1$, $i = 2m-1$ and $j = 2m-2$, …, $i = d+1$ and $j = d$ (all those choices of j are possible since $i > d$), we see that

$$\pi(m) < \pi(m-1) < \cdots < \pi(d) < d < d+1 < \cdots < 2m.$$

This implies in particular that $d \geq m$. Thus, we see that $\pi(d) = d-1$, and thus the face at the left of h_d is incident to exactly one half-edge. The claim is shown. □

This approach generalizes to the case where one of the matrices, say X, is of low rank. The following result is essentially a restatement in the case of two matrices of a result obtained by Collins, Hasebe, and Sakuma [CHS18], in the study of cyclical monotone independence.

Theorem 3.4.1. *Let (X^N) and (Y^N) be two sequences of deterministic matrices, with X^N and Y^N of size $N \times N$, and U^N be a Haar-distributed unitary matrix of size $N \times N$. Assume that there exists $0 \leq \eta < 1$ and a sequence of real numbers $(\tau_X(x^k))_{k\geq 1}$ such that for all $k \geq 1$,*

$$N^{-\eta}\operatorname{Tr}\left(X^k\right) \to \tau_X(x^k) \text{ as } N \to \infty,$$

and that (Y^N) converges in distribution, with limiting distribution τ_Y. Then, for all monomial $P \in \mathbb{C}\langle x, y\rangle$, written

$$P = x^{k_1}y^{l_1}\cdots x^{k_m}y^{l_m},$$

with $k_1, l_1, \dots, k_m, l_m \geq 1$, we have

$$N^{-\eta}\mathbb{E}\left[\operatorname{Tr}(P)(X, UYU^*)\right] \to \tau_X(x^{\sum_{i=1}^m k_i})\prod_{i=1}^m \tau_Y(y^{l_i}) \text{ as } N \to \infty.$$

Proof. It suffices to consider a balanced monomial P of degree $2m$, written

$$P = x^{k_1}uy^{l_1}u^*\cdots x^{k_m}uy^{l_m}u^* \in \mathbb{C}\langle x, y, u, u^*\rangle,$$

with $k_1, l_1, \dots, k_m, l_m \geq 1$. By Proposition 3.3.27, the leading order of the expectation of $N^{-\eta}\mathbb{E}\operatorname{Tr} P$ is

$$\begin{aligned} N^{-\eta}\mathbb{E}\left[\operatorname{Tr}(P)\right] &= N^{1-\eta}\mathcal{W}_{0,1}^{(0),N}(P) + \mathcal{O}\big(N^{-1-\eta}\big) \\ &= (-1)^{m+1}\sum_{\substack{\mathfrak{m}\in\mathfrak{C}(0,[2m],\boldsymbol{\epsilon_P},\gamma_{\boldsymbol{P}})\\ \mathfrak{m}\text{ is connected}}} (-1)^{c(\phi_\mathfrak{m})}N^{1-\eta}\operatorname{tr}_{\phi_\mathfrak{m}^+}(\boldsymbol{X})\operatorname{tr}_{\phi_\mathfrak{m}^-}(\boldsymbol{Y}) + \mathcal{O}\big(N^{-1-\eta}\big), \end{aligned}$$

with

$$\boldsymbol{X} = X^{k_1}\otimes\cdots\otimes X^{k_m} \quad \text{and} \quad \boldsymbol{Y} = Y^{l_1}\otimes\cdots\otimes Y^{l_m}.$$

Consider $\mathfrak{m} \in \mathfrak{C}(0, [2m], \boldsymbol{\epsilon_P}, \gamma_{\boldsymbol{P}})$. The associated weight is

$$\operatorname{tr}_{\phi_\mathfrak{m}^+}(\boldsymbol{X})\operatorname{tr}_{\phi_\mathfrak{m}^-}(\boldsymbol{Y}).$$

We notice that in the large N limit, $\mathrm{tr}_{\phi_{\mathfrak{m}}^{-}}(\boldsymbol{Y})$ is of constant order, while $\mathrm{tr}_{\phi_{\mathfrak{m}}^{+}}(\boldsymbol{X})$ is of order $N^{-c(\phi_{\mathfrak{m}}^{+})(1-\eta)}$. This means that the leading order of $N^{-\eta}\mathbb{E}\,\mathrm{Tr}\,P$ involves only maps with $c(\phi_{\mathfrak{m}}^{+})=1$. By planarity, this is only possible if $\phi_{\mathfrak{m}}^{-}$ is the identity. Hence,

$$N^{1-\eta}\mathcal{W}_{0,1}^{(0),N}(P)=N^{-\eta}\,\mathrm{Tr}\Big(X^{\sum_i k_i}\Big)\prod_{j=1}^{m}\mathrm{tr}\Big(Y^{l_j}\Big)+\mathcal{O}\big(N^{\eta-1}\big).$$

Taking the limit $N\to\infty$ yields the result. □

Remark 3.4.2. We may wonder how the previous Theorem transforms when considering general joint moments, i.e. expectation of product of traces. For the same reason as in the proof, a map $\mathfrak{m}$ describing the leading order will have a permutation $\phi_{\mathfrak{m}}^{+}$ satisfying $c(\phi_{\mathfrak{m}}^{+})=1$. One might then guess that such a $\mathfrak{m}$ will be a cactus: a map in which two simple cycles have at most one vertex in common. This point is left for further study.

3.5 Tutte-like equations

We will now state induction relations that applies to the sums of maps $\mathcal{M}_{0,l}^{(g),N}$ defined in Definition 3.3.29. They are obtained by a procedure very similar to the one used by Tutte in [Tut68]. These induction relations are the analog of the topological recursion for matrices of the GUE [EO08]. Similar induction relations have been obtained for maps related to the Gaussian case in [GM07] and [Mau06], and for maps with “dotted edges” in the unitary case for $g=0$ in [CGM09]. More precisely, we will prove the following theorem.

Theorem 3.5.1. *Let $N\in\mathbf{N}^*$, $\boldsymbol{P}=(P_1,\dots,P_lu)\in\mathcal{X}^l$ be monomials. Then, for $g\geq 0$ and $m=\frac{1}{2}\deg\boldsymbol{P}\geq 2$, we have the induction relation*

$$\begin{aligned}
\mathcal{M}_{0,l}^{(g),N}(P_1,\dots,P_lu)=-\sum_{P_l=QuR}&\Big[\mathcal{M}_{0,l+1}^{(g-1),N}(P_1,\dots,P_{l-1},Qu,Ru)\\
&+\sum_{\substack{g_1+g_2=g\\ I\subset[l-1]}}\mathcal{M}_{0,|I|+1}^{(g_1),N}(\boldsymbol{P}|_I,Qu)\mathcal{M}_{0,l-|I|}^{(g_2),N}(\boldsymbol{P}|_{I^c},Ru)\\
&+\sum_{j=1}^{l-1}\mathcal{M}_{0,l-1}^{(g),N}(P_1,\dots,P_{j-1},P_{j+1},\dots,P_{l-1},RQuP_lu)\Big]\\
+\sum_{P_l=Qu^*R}&\Big[\mathcal{M}_{0,l+1}^{(g-1),N}(P_1,\dots,P_{l-1},Q,R)\\
&+\sum_{\substack{g_1+g_2=g\\ I\subset[l-1]}}\mathcal{M}_{0,|I|+1}^{(g_1),N}(\boldsymbol{P}|_I,Q)\mathcal{M}_{0,l-|I|}^{(g_2),N}(\boldsymbol{P}|_{I^c},R)\\
&+\sum_{j=1}^{l-1}\mathcal{M}_{0,l-1}^{(g),N}(P_1,\dots,P_{j-1},P_{j+1},\dots,P_{l-1},RQP_l)\Big],
\end{aligned}$$

where we use the notation $\boldsymbol{P}|_I=(P_i)_{i\in I}$ and set by convention, $\mathcal{M}_{0,l}^{(-1),N}=0$ and $\mathcal{M}_{0,0}^{(g),N}=0$.

With a similar proof, we can state a similar theorem for $\boldsymbol{P}=(P_1,\dots,P_lu^*)$. Equivalently, this is a consequence of the invariance by conjugation of the sums of maps, see Lemma 3.3.31.

Theorem 3.5.1 describes how a map of unitary type can be decomposed into one or several maps. In the Section 3.5.1, we will describe the precise procedure used to cut maps of unitary type into one or more maps of unitary types, or equivalently to decompose the permutations describing a particular map. The procedure can be understood as an elaborate way of contracting an edge in a map. This allows us to give an interpretation to the terms appearing in the recurrence.

Consider first the terms in the last two lines (the other terms have a similar interpretation). When contracting an edge, three situations can occur.

- The edge is a non-contractible loop, informally it goes around a handle of the surface, in that case when contracting it we remove the handle and create a new vertex, this corresponds to the term

$$\sum_{P_l=Qu^*R} \mathcal{M}^{(g-1),N}_{0,l+1}(P_1,\dots,P_{l-1},Q,R)\,.$$

- The edge is a contractible loop. When contracting the edge we cut the map into two disconnected components. This corresponds to the term

$$\sum_{P_l=Qu^*R}\sum_{\substack{g_1+g_2=g\\ I\subset[l-1]}} \mathcal{M}^{(g_1),N}_{0,|I|+1}(\boldsymbol{P}|_I,Q)\mathcal{M}^{(g_2),N}_{0,l-|I|}(\boldsymbol{P}|_{I^c},R)\,.$$

- The edge is not a loop. When contracting the edge we just merge two vertices. This corresponds to the term

$$\sum_{j=1}^{l-1}\sum_{P_j=Qu^*R} \mathcal{M}^{(g),N}_{0,l-1}(P_1,\dots,P_{j-1},P_{j+1},\dots,P_{l-1},RQP_l)\,.$$

The other terms have a similar form. They do not correspond to the contraction of an edge, but rather to the erasure of a black vertex. This Theorem shows that a sum of maps can be expressed in terms of the sums of maps with either lower genus, lower overall degree, or lower number of vertices.

Before describing precisely the procedure used to cuts the maps of unitary type, and giving the proof of Theorem 3.5.1, we show how to rewrite these equation in a compact way. The function $\mathcal{M}^{(g),N}_{0,l}$ was defined only on some monomials, i.e. on the set $\mathcal{X}^l$. Recall that $\mathcal{A}$ is the algebra of non-commutative polynomials in the variables $u, u^*, a_1, a_1^*, \dots$. We now extend $\mathcal{M}^{(g),N}_{0,l}$ by linearity to an linear form on the tensor space $\mathcal{A}^{\otimes l} = \mathcal{A}\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}}\mathcal{A}$. We will use interchangeably the notation $\mathcal{M}^{(g),N}_{0,l}(P_1,\dots,P_l)$ ($M^{(g),N}_{0,l}$ as a multilinear function) and $\mathcal{M}^{(g),N}_{0,l}(P_1\otimes\cdots\otimes P_l)$ ($M^{(g),N}_{0,l}$ as a linear function on $\mathcal{A}^{\otimes l}$). We can then rewrite Theorem 3.5.1 using the notion of non-commutative derivative.

Let $\boldsymbol{P}=(P_1,\dots,P_l)$ be a k-tuple of polynomials, and $I=\{i_1<i_2<\cdots<i_p\}$ be a non-empty subset of $[l]$, then we define

$$\boldsymbol{P}_I = P_{i_1}\otimes P_{i_2}\otimes\cdots\otimes P_{i_p}.$$

We define the operations $\times$ and $\sharp$ as follows. Let $\boldsymbol{P}\in\mathcal{A}^l$, $I\subset[l-1]$, $Q=Q_1\otimes Q_2\in\mathcal{A}^{\otimes 2}$ and $S=S_1\otimes S_2\in\mathcal{A}^{\otimes 2}$, then we set

$$Q\times S = (Q_1\otimes Q_2)\times(S_1\otimes S_2) = (Q_1S_1)\otimes(Q_2S_2)\,, \tag{3.17}$$

and

$$\boldsymbol{P}_I\otimes\boldsymbol{P}_{I^c}\sharp Q = \boldsymbol{P}_I\otimes Q_1\otimes\boldsymbol{P}_{I^c}\otimes Q_2. \tag{3.18}$$

Definition 3.5.2. *The* ***logarithmic non-commutative derivative*** $\partial\colon\mathcal{A}\to\mathcal{A}^{\otimes 2}$ *with respect to* u *of a monomial* $P\in\mathcal{A}$ *is defined by*

$$\partial P = \sum_{P=QuR} Qu\otimes R - \sum_{P=Qu^{-1}R} Q\otimes u^{-1}R\in\mathcal{A}^{\otimes 2}.$$

The definition extends by linearity to any polynomial in $\mathcal{A}$.

Remark 3.5.3. This derivative was introduced by Voiculescu in [Voi99, Section 8.1]. However, as remarked by an anonymous reviewer, this corresponds to the derivation on the unitary group invariant under multiplication from the right. Near the identity, this derivation corresponds to the ordinary gradient ∇_H where $\exp(H) = U$, hence the name logarithmic derivative.

Remark 3.5.4. Consider a monomial $P \in \mathcal{A}$ evaluated in $U, U^*, A_1, A_1^*, \ldots$, and denote by $\partial_{m,j}$ the derivative with respect to the coefficient (m, j) of U. We have

$$\sum_m U_{m,k}(\partial_{m,j}P)_{i,l} = \sum_m \sum_{P=QUR} Q_{i,m}U_{m,k}R_{j,l} = \sum_{P=QUR} (QU)_{i,k}R_{j,l}\,,$$

and similarly, if $\overline{\partial}_{m,j}$ is the derivative with respect to the coefficient (m, j) of U^*

$$\sum_m U^*_{j,m}(\overline{\partial}_{k,m}P)_{i,l} = \sum_m \sum_{P=QU^*R} Q_{i,k}U^*_{j,m}R_{m,l} = \sum_{P=QUR} Q_{i,k}(U^*R)_{j,l}\,.$$

In coordinates, the non-commutative derivative thus corresponds to

$$(\partial P)_{\substack{i,k\\j,l}} = \sum_m \left(U_{m,k}\partial_{m,j}P - U^*_{j,m}\overline{\partial}_{k,m}P\right)_{i,l}\,.$$

Definition 3.5.5. *The* ***cyclic derivative*** $\mathcal{D}\colon \mathcal{A} \to \mathcal{A}$ *with respect to* u *of a monomial* $P \in \mathcal{A}$ *is defined by*

$$\mathcal{D}P = \sum_{P=QuR} RQu - \sum_{P=Qu^{-1}R} u^{-1}RQ.$$

The definition extends by linearity to any polynomial in $\mathcal{A}$*.*

Remark 3.5.6. Let P be a monomial. In coordinates, the cyclic derivative is

$$(\mathcal{D}P)_{i,j} = \sum_{k,m} \left(U_{m,j}\partial_{m,i}P - U^*_{m,j}\overline{\partial}_{j,m}P\right)_{k,k} = \sum_k (\partial P)_{\substack{j,k\\i,k}}\,.$$

To take advantage of the notation just introduced, we shall write

$$\mathcal{M}^{(g),N}_{0,l} \otimes \mathcal{M}^{(g'),N}_{0,l'}(P_1 \otimes \cdots \otimes P_{l+l'})$$

with $P_i \in \mathcal{A}$ to mean

$$\mathcal{M}^{(g),N}_{0,l} \otimes \mathcal{M}^{(g'),N}_{0,l'}(P_1 \otimes \cdots \otimes P_{l+l'}) = \mathcal{M}^{(g),N}_{0,l}(P_1, \ldots, P_l)\mathcal{M}^{(g'),N}_{0,l'}(P_{l+1}, \ldots, P_{l+l'})\,.$$

Theorem 3.5.1 can then be rewritten as follows.

Corollary 3.5.7. *For* $m \geq 2$*,* $g \geq 1$*, and* $P_1, \ldots, P_l \in \mathcal{A}$*, we have the following equation*

$$\begin{aligned}\sum_{I\subset[l-1]} \sum_{g=g_1+g_2} &\mathcal{M}^{(g_1),N}_{0,|I|+1} \otimes \mathcal{M}^{(g_2),N}_{0,|I^c|+1}(\boldsymbol{P}_I \otimes \boldsymbol{P}_{I^c}\sharp\partial P_l)\\ &= -\mathcal{M}^{(g-1),N}_{0,l+1}(P_1 \otimes \cdots \otimes P_{l-1} \otimes \partial P_l) - \sum_{j=1}^{l-1} \mathcal{M}^{(g),N}_{0,l-1}(P_1 \otimes \cdots \otimes \check{P}_j \otimes \cdots \otimes (\mathcal{D}P_j)P_l),\end{aligned}$$

where $\check{P}_j$ *means that the factor* P_j *is omitted.*

Proof. We group the terms two by two. We have

$$\begin{aligned}
&-\sum_{P_l=QuR}\mathcal{M}^{(g-1),N}_{0,l+1}(P_1,\dots,P_{l-1},Qu,Ru)+\sum_{P_l=Qu^*R}\mathcal{M}^{(g-1),N}_{0,l+1}(P_1,\dots,P_{l-1},Q,R)\\
&\qquad=-\mathcal{M}^{(g-1),N}_{0,l+1}(P_1,\dots,P_{l-1},\partial(P_lu))+\mathcal{M}^{(g-1),N}_{0,l+1}(P_1,\dots,P_{l-1},P_l,1)\\
&\qquad=-\mathcal{M}^{(g-1),N}_{0,l+1}(P_1,\dots,P_{l-1},\partial(P_lu))\,.
\end{aligned}$$

where we used the traciality property of Lemma 3.3.31 to replace u^*Ru by R in the last argument of $\mathcal{M}^{(g-1),N}_{0,l+1}$, in the second line. In the third line we used the fact that there are no connected map with $l+1$ vertices, $l\geq 1$, and one vertex of degree 0.

We proceed similarly for the two other terms. We have

$$\begin{aligned}
&-\sum_{P_l=QuR}\sum_{\substack{g_1+g_2=g\\ I\subset[l-1]}}\mathcal{M}^{(g_1),N}_{0,|I|+1}(\boldsymbol{P}|_I,Qu)\mathcal{M}^{(g_2),N}_{0,l-|I|}(\boldsymbol{P}|_{I^c},Ru)\\
&+\sum_{P_l=Qu^*R}\sum_{\substack{g_1+g_2=g\\ I\subset[l-1]}}\mathcal{M}^{(g_1),N}_{0,|I|+1}(\boldsymbol{P}|_I,Q)\mathcal{M}^{(g_2),N}_{0,l-|I|}(\boldsymbol{P}|_{I^c},R)\\
&\qquad=-\sum_{\substack{g_1+g_2=g\\ I\subset[l-1]}}\mathcal{M}^{(g_1),N}_{0,|I|+1}\otimes\mathcal{M}^{(g_2),N}_{0,l-|I|}(\boldsymbol{P}|_I\otimes\boldsymbol{P}|_{I^c}\sharp(\partial Pu))+\mathcal{M}^{(g),N}_{0,l}(\boldsymbol{P})\,,
\end{aligned}$$

where we used that there are no maps with one vertex of degree 0 if $g\geq 1$ or $l\geq 2$.

The third term is

$$\begin{aligned}
&-\sum_{j=1}^{l-1}\sum_{P_j=QuR}\mathcal{M}^{(g),N}_{0,l-1}(P_1,\dots,P_{j-1},P_{j+1},\dots,P_{l-1},RQuP_lu)\\
&+\sum_{j=1}^{l-1}\sum_{P_j=Qu^*R}\mathcal{M}^{(g),N}_{0,l-1}(P_1,\dots,P_{j-1},P_{j+1},\dots,P_{l-1},RQP_l)\\
&\qquad=-\sum_{j=1}^{l-1}\mathcal{M}^{(g),N}_{0,l-1}(P_1,\dots,P_{j-1},P_{j+1},\dots,P_{l-1},(\mathcal{D}P_j)P_lu)\,.
\end{aligned}$$

Putting the three terms together, we get the result when P_l is a monomial that finishes by u. The cyclicity property of Lemma 3.3.31 implies that it is then the same if P_l contains a u. Finally, if P_l does not contains a u, then it is either a constant and the formula is clear, or it contains a u^*. In the latter case, the conjugation property of Lemma 3.3.31 allows us to recover the result.

Finally, the result extend by linearity to all polynomials, as it is linear in each of the P_i. □

Notice that the formula in Corollary 3.5.7 is valid not only for monomials P_l finishing by a u but also for all polynomials.

3.5.1 How to cut maps

In this section, we fix two integers $m\geq 2$ and $g\geq 0$, a permutation $\gamma\in\mathfrak{S}_{2m}$ and $\boldsymbol{\epsilon}\in\mathcal{E}_{2m}=\{\boldsymbol{\epsilon}=(\epsilon(i))_{i\in[2m]}\in\{\pm1\}^{2m}\colon\sum_{i=1}^{2m}\epsilon(i)=0\}$. By the cyclicity and the symmetry properties proved in Lemma 3.3.31, we can assume that the polynomial P_l finishes by the letter u. We thus assume that $\epsilon(2m)=+1$. We consider a map of unitary type, $\mathfrak{m}\in\mathfrak{C}(g,m,\gamma,\boldsymbol{\epsilon})$, with r black vertices. Let S denote the underlying surface of the map $\mathfrak{m}$.

By Theorem 3.3.22, this map is described by the permutation $\pi:=\pi_{\mathfrak{m}}$ and the tuple of transpositions $\tau_{\mathfrak{m}}=(\tau_1,\dots,\tau_r)\in\overrightarrow{\mathcal{W}}^r(\pi^{(\epsilon)})$, with r related to g, $\pi_{\mathfrak{m}}$ and $\gamma_{\mathfrak{m}}$ by Euler's formula, see (3.13). We will consider two ways of cutting this map, depending on whether $\tau_r(2m)=2m$ or not.

Remark 3.5.8. We can also see vertices as “holes” in the surface, that is, we take the underlying surface S to be a surface with boundaries. A vertex is then a boundary component (homeomorphic to a circle) of the surface S. An edge is then a path connecting two boundary components. See Figure 3.10 for an example.

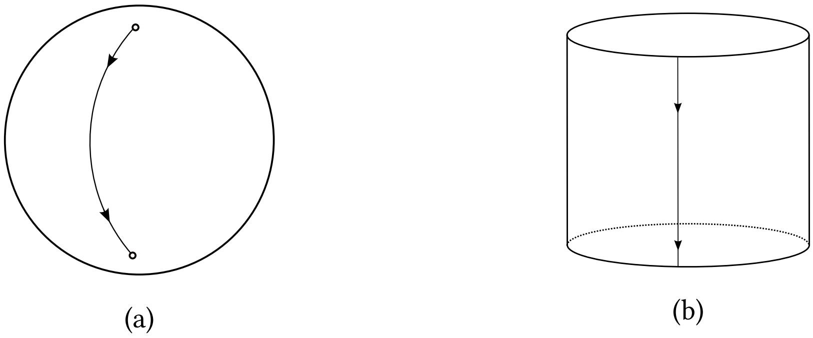


Figure 3.10: (a) An oriented map of genus 0 with two vertices (b) The corresponding map where vertices are seen as boundary components. The underlying surface is a cylinder.

We will see white vertices as boundary components of the underlying surface, as explained in Remark 3.5.8. To describe the cutting procedure, we introduce the definition of a corner, see Figure 3.11.

Definition 3.5.9. *Let $\mathfrak{m}$ be a map of unitary type. Consider a (white or black) vertex v in $\mathfrak{m}$, seen as a boundary component of a surface, as explained in Remark 3.5.8, and a half-edge h.*

*The **corner** at the left (respectively at the right) of h is the connected, closed set C such that*

- *C is a subset of the boundary component B corresponding to v;*
- *C is in the boundary of the face f at the left (respectively right) of the half-edge h;*
- *the boundary of C is $\{u, v\}$, where u is the point of intersection of B and the half-edge h, and v is the point of intersection of B and the half-edge h' that follows the half-edge h when going around the face f in the clockwise (resp. counterclockwise) direction starting from the left (resp. the right) of h.*

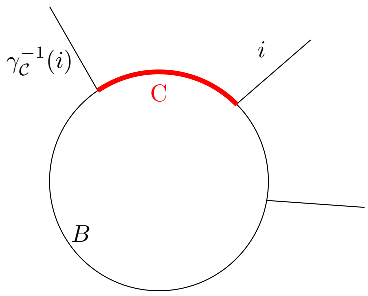


Figure 3.11: The corner C at the left of i, displayed with a red thick line. Here, i is a white half-edge.

First case: $\tau_r(2m) = 2m$

In this case, $\mathrm{val}(\tau_r) < 2m$, see Definition 3.2.10. The monotonicity of the walk $\boldsymbol{\tau}$ implies that for all $i \in [r]$, $\mathrm{val}(\tau_i) < 2m$, and in particular $\tau_i(2m) = 2m$. By Definition 3.3.16, the half-edge $2m$ is not connected to any black half-edge, and is thus connected to a white-half-edge, say the j-th one. Note that by our assumption that $\epsilon(2m) = +1$, $\epsilon(j) = -1$. Notice that because $\mathfrak{m}$ is non-decreasing, $\tau_r(2m) = 2m$ implies that for all i, $\tau_i(2m) = 2m$.

We construct a map $\mathfrak{m}'$ of unitary type from $\mathfrak{m}$ using the following procedure, depicted in Figure 3.12.

1. We choose a path η in the face f at the right of the half-edge $2m$. This path is chosen to start from the white vertex w_{2m}, attached in the corner (see Definition 3.5.9) at the right of the half-edge $2m$, and end at w_j, attached in the corner at the left of the half-edge j. As faces are homeomorphic to disks, there is only one way to choose η up to homotopy.

2. We remove the edge containing the half-edges j and $2m$.

3. We cut the surface along η. Depending on the cases we connect two distinct boundary components of S, or we connect one boundary component to itself.

Remark 3.5.10. Notice that if w_j and w_{2m} are distinct vertices, this surgery is the usual contraction of an edge.

Remark 3.5.11. If the path η is a loop, then it may be that the resulting map $\mathfrak{m}'$ is disconnected. Furthermore, one of the connected component can be a vertex with no edges. In this case, we remove this component.

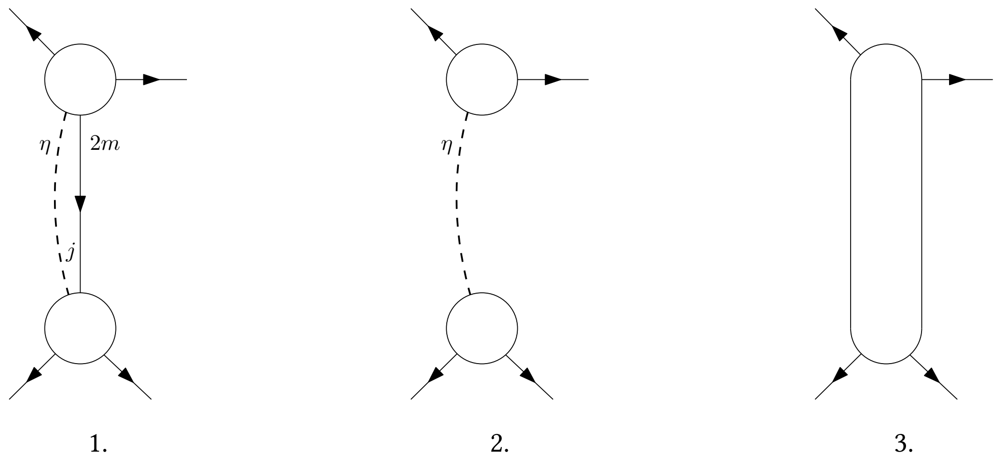


Figure 3.12: First way to cut the map.

Lemma 3.5.12. *If $\mathfrak{m}$ is a map of unitary type with label set I of size $2m$, then $\mathfrak{m}'$ is a map of unitary type, with label set $I \setminus \{j, 2m\}$, of size $2m-2$. Furthermore, if $\mathfrak{m}$ is non-decreasing, then so is $\mathfrak{m}'$.*

Proof. We first check that $\mathfrak{m}'$ is a map. We only need to check that each face of $\mathfrak{m}'$ is homeomorphic to a disk. We only modify the faces f_{left} and f_{right} at the left and the right of the edge (w_{2m}, w_j). At step 2, when we remove the edge, we connect f_{left} and f_{right}. However, at step 3, we cut along a path homotopic to the edge, thus separating the two faces. All the faces of $\mathfrak{m}'$ thus remain disks.

We now check that this map is a map of unitary type. The map $\mathfrak{m}'$ has $2m-2$ labelled white half-edges (maybe 0 if $m = 1$), with label set $I \setminus \{j, 2m\}$ (property 2 in Definition 3.3.7 is satisfied). The black vertices have not been modified when transforming $\mathfrak{m}$ into $\mathfrak{m}'$, so properties 1 and 3 in Definition 3.3.7 are satisfied. As the black vertices are not modified, a non-decreasing map remain non-decreasing. □

Let us now compute the permutations that represent $\mathfrak{m}'$. We will need the notion of the trace of a permutation introduced by Kreweras [Kre72]. This notion has nothing to do with the notion of trace of a matrix.

Definition 3.5.13. *Let A be a finite subset of $\mathbf{N}$. The **trace** of a permutation $\sigma \in \mathfrak{S}(A)$, on $B \subset A$, denoted by $\operatorname{Tr}(\sigma; B)$, is the permutation in $\mathfrak{S}(B)$ defined for each $x \in B$ by*

$$\operatorname{Tr}(\sigma; B)(x) = \sigma^{p_x}(x),$$

with $p_x \geq 1$ the smallest integer so that $\sigma^p(x) \in B$.

Computing the trace of a permutation in cycle notation is straightforward: write the cycle decomposition of σ, and erase all elements in the cycles that do not belong to B.

Let $I_j = [2m-1] \setminus \{j\}$.

Lemma 3.5.14. *Let* $\pi' := \mathrm{Tr}(\pi_{\mathfrak{m}}; I_j)$. *We have* $\pi' = \pi_{\mathfrak{m}}(j\,2m) = (j\,2m)\pi_{\mathfrak{m}}$, *and* $\pi_{\mathfrak{m}'} = \pi' = \pi_{\mathfrak{m}}|_{I_j}$.

Proof. We have assumed that the half-edges $2m$ and j are connected to form an edge. This imply $\pi_{\mathfrak{m}}(2m) = j$ and $\pi_{\mathfrak{m}}(j) = 2m$. Thus, as $\pi_{\mathfrak{m}}$ is a permutation, for all $k \in I_j$, $\pi_{\mathfrak{m}}(k) \in I_j$. This means that in the notation of Definition 3.5.13, $p_k = 1$. We get the first claim, and that $\pi' = \pi_{\mathfrak{m}}|_{I_j}$.

When removing the edge at step 2, it is clear that the map we obtain is still described by $\pi_{\mathfrak{m}}$, with a cycle removed. When cutting the map at step 3, we do not modify the edges further. □

Lemma 3.5.15. *Let* $\gamma' := \mathrm{Tr}(\gamma(j\,2m); I_j)$. *We have* $\gamma_{\mathfrak{m}'} = \gamma'$.

Proof. Assume first that w_j and w_{2m} are two distinct vertices. Let $c = (u_1\,\ldots\,u_p\,j)$ and $c' = (u'_1\,\ldots\,u'_{p'}\,2m)$ be the cycles that represent them. After cutting the map at step 3, the vertices are replaced by a vertex with structure $(u_1\,\ldots\,u_p\,u'_1\,\ldots\,u'_{p'}) = \mathrm{Tr}(cc'(j\,2m); I_j)$.

If $w_j = w_{2m}$, this vertex is represented by a cycle $c = (u_1\,\ldots\,u_p\,j\,u'_1\,\ldots\,u'_{p'}\,2m)$, which we cut using the transposition $(j\,2m)$. We obtain two vertices represented by the two cycles $\mathrm{Tr}(c(j\,2m); I_j)$. □

Lemma 3.5.16. *We have* $\phi_{\mathfrak{m}'} = \mathrm{Tr}(\phi_{\mathfrak{m}}; I_j)$.

Proof. By Lemmas 3.3.19, 3.5.14 and 3.5.15,

$$\phi_{\mathfrak{m}'} = \gamma'\pi'^{-1} = \mathrm{Tr}(\gamma_{\mathfrak{m}}(j\,2m); I_j)\pi_{\mathfrak{m}}|_{I_j}^{-1}.$$

Notice first that for any $k \in I$ and $p \in \mathbf{N}$, $(\gamma_{\mathfrak{m}}\pi_{\mathfrak{m}}^{-1})^p(k) \in \{j, 2m\}$ if and only if

$$((j\,2m)\gamma_{\mathfrak{m}}\pi_{\mathfrak{m}}^{-1}(j\,2m))^p(k) \in \{j, 2m\}.$$

This implies that

$$\mathrm{Tr}\big(\gamma_{\mathfrak{m}}\pi_{\mathfrak{m}}^{-1}; I_j\big) = \mathrm{Tr}\big((j\,2m)\gamma_{\mathfrak{m}}\pi_{\mathfrak{m}}^{-1}(j\,2m); I_j\big) = \mathrm{Tr}\big((j\,2m)\gamma_{\mathfrak{m}}^{-1}\pi'^{-1}; I_j\big),$$

were we used Lemma 3.5.14 for the last equality.

Then, as $\pi'(j) = j$ and $\pi'(2m) = 2m$, we have

$$\mathrm{Tr}\big((j\,2m)\gamma\pi'^{-1}; I_j\big) = \mathrm{Tr}((j\,2m)\gamma; I_j)\pi'^{-1} = \phi_{\mathfrak{m}'}.$$

□

Lemma 3.5.17. *If the map* $\mathfrak{m}$ *of unitary type is connected, then the map* $\mathfrak{m}'$ *has one or two connected components. Furthermore, if* j *and* $2m$ *do not belong to the same cycle in* γ, $\mathfrak{m}'$ *is connected.*

Proof. Assume first that j and $2m$ belong to the same cycle in γ. This means that $w_j = w_{2m}$. If we erase the edge containing the half-edges $2m$ and j, $\mathfrak{m}$ stays connected. However, when we cut the map along the path η, we may separate the map into two connected components. More precisely, we separate the map into two connected components if and only if η is homologically trivial, that is, the boundary of a surface embedded in S.

If j and $2m$ belong to different cycles, that is $w_j \neq w_{2m}$, then when removing the edge we may disconnect the two vertices but we then merge them. Consequently, the map $\mathfrak{m}'$ is connected. □

Using the permutations $\gamma' = \gamma_{\mathfrak{m}'}$ and $\phi_{\mathfrak{m}}$, and Lemma 3.5.17, we can now compute the genus of $\mathfrak{m}'$. We recall Euler's formula (3.13) for a map of genus $g_{\mathfrak{m}}$ with $C_{\mathfrak{m}}$ connected components

$$2C_{\mathfrak{m}} - 2g_{\mathfrak{m}} = c(\gamma_{\mathfrak{m}}) + c(\phi_{\mathfrak{m}}) - m - r.$$

We now discuss the several cases that can occur. First, if both j and $2m$ are fixed points of $\phi_{\mathfrak{m}}$ then it means that $\mathfrak{m}$ is reduced to a vertex with 2 half-edges. We will assume in what follows that $m \geq 2$.

If j (respectively $2m$) is a fixed point of $\phi_{\mathfrak{m}} = \gamma_{\mathfrak{m}}\pi_{\mathfrak{m}}^{-1}$, then it means in terms of map that the face at the left of the white half-edge j (resp. $2m$) is at the left of the half-edge j (resp. $2m$) only. In term of permutations, we have $\gamma_{\mathfrak{m}}(2m) = \phi_{\mathfrak{m}}\pi_{\mathfrak{m}}(2m) = j$ (resp. $\gamma_{\mathfrak{m}}(j) = 2m$). Then, $(j\,2m)\gamma_{\mathfrak{m}}(2m) = 2m$ (resp. $(j\,2m)\gamma_{\mathfrak{m}}(j) = j$), and when taking the trace on I_j, we remove one cycle of $(j\,2m)\gamma_{\mathfrak{m}}$. Furthermore, if we remove the disk corresponding to the face at the left of the half-edge j (resp. $2m$), the resulting map $\mathfrak{m}'$ is connected.

If j and $2m$ are not fixed points of $\phi_{\mathfrak{m}}$, then none of the connected component is reduced to a vertex without edges. It implies that the total number of faces and cycles of the associated permutation stays the same. We have $c(\phi_{\mathfrak{m}}) = c(\phi_{\mathfrak{m}'})$. The resulting map $\mathfrak{m}'$ has one or two connected components: by cutting the map we either remove a handle (and decreased the genus by one) or disconnected the map.

It gives us one particular case (the degenerate case were one connected component is reduced to a vertex without edges):

1. if j or $2m$ is a fixed point of $\phi_{\mathfrak{m}}$, then $c(\phi_{\mathfrak{m}'}) = c(\phi_{\mathfrak{m}}) - 1$, $c(\gamma') = c(\gamma)$, and $\mathfrak{m}'$ is connected. Thus, $g_{\mathfrak{m}'} = g_{\mathfrak{m}}$.

If both j and $2m$ are not fixed points of $\phi_{\mathfrak{m}}$, we have the three cases.

2. If j and $2m$ belong to the same cycle of $\gamma_{\mathfrak{m}}$, and $\mathfrak{m}'$ is connected, then $c(\gamma_{\mathfrak{m}'}) = c(\gamma_{\mathfrak{m}}) + 1$, $c(\phi_{\mathfrak{m}'}) = c(\phi_{\mathfrak{m}})$, and $g' = g - 1$ (by Euler formula).

3. If j and $2m$ belong to the same cycle of $\gamma_{\mathfrak{m}}$, and $\mathfrak{m}'$ has two connected components, then $c(\gamma_{\mathfrak{m}'}) = c(\gamma_{\mathfrak{m}}) + 1$, $c(\phi_{\mathfrak{m}'}) = c(\phi_{\mathfrak{m}})$, and $g' = g$.

4. If j and $2m$ belong to two different cycles of $\gamma_{\mathfrak{m}}$, then $c(\gamma_{\mathfrak{m}'}) = c(\gamma_{\mathfrak{m}}) - 1$, $c(\phi_{\mathfrak{m}'}) = c(\phi_{\mathfrak{m}})$, and $g' = g$.

Second case: $\tau_r(2m) \neq 2m$

In this case, the white half-edge labelled $2m$ is connected to a black vertex. Let $j = \tau_r(2m) \in \epsilon^{-1}(+1)$ (as the support of the transpositions τ_i is contained in $\epsilon^{-1}(+1)$). In that case, the last black vertex has an outgoing half-edge labelled by $2m$ by Lemma 3.3.12. Similarly, the last black vertex has an outgoing half-edge labelled j.

We construct a unitary map $\mathfrak{m}'$ from $\mathfrak{m}$ using the following procedure, depicted in Figure 3.13. Notice that the first step is possible as there are no loop with black edges, as indicated in Remark 3.3.10.

1. We choose two paths η_1 and η_2 contained respectively in the face f_{2m} at the left of the half-edge $2m$, and f_j at the left of the half-edge j. The path η_1 (respectively η_2) is chosen to start from the white vertex w_{2m} (respectively w_j), attached in the corner at the left of the half-edge $2m$ (respectively j), and end at the r-th black vertex, attached in the corner at the left of the half-edge labelled $2m$ (respectively j).

2. We remove the r-th black vertex, and attach each ingoing edge to the outgoing edge that follows it in the counterclockwise order.

3. We cut the surface S along $\eta = \eta_1 \cup \eta_2$.

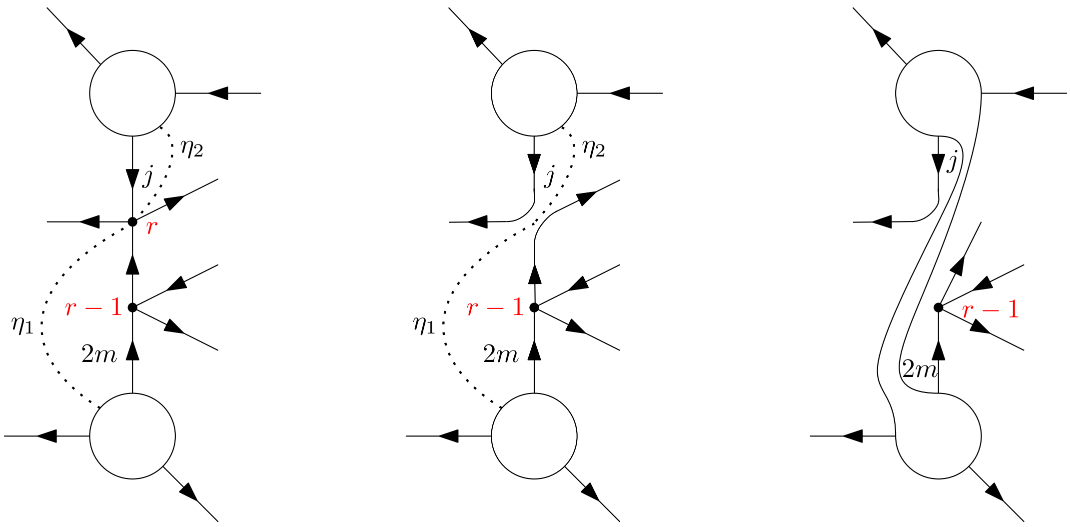


Figure 3.13: Second way to cut the map.

Lemma 3.5.18. *If $\mathfrak{m}$ is a map of unitary type with labelling set I and with r black vertices, then $\mathfrak{m}'$ is a map of unitary type with labelling set I and with $r-1$ black vertices. Furthermore, if $\mathfrak{m}$ is non-decreasing, then so is $\mathfrak{m}'$.*

Proof. As in the proof of Lemma 3.5.12, we first prove that $\mathfrak{m}'$ is a map. We have to check that the faces are homeomorphic to disks. When removing the black vertex, at step 2, we may have connected two faces together, or may have connected a face to itself, thus creating a "face" homeomorphic to an annulus. However, when we cut the map, at step 3, we recover one or two faces homeomorphic to disks. Thus, $\mathfrak{m}'$ is a map.

We now show that $\mathfrak{m}'$ is indeed a map of unitary type. The map $\mathfrak{m}'$ has $r-1$ black vertices. We removed the last black vertex and did not create any new edge linking two black vertices, thus properties 1 and 3 of Definition 3.3.7 are satisfied. We did not remove any white half-edge so property 2 is satisfied as well. Thus, $\mathfrak{m}'$ is of unitary type. Furthermore, as we removed the last black vertex and let the other ones unchanged, if $\mathfrak{m}$ is non-decreasing (recall Definition 3.3.14), $\mathfrak{m}'$ is non-decreasing as well. □

Let us now compute the permutations that represent $\mathfrak{m}'$.

Lemma 3.5.19. *Let $\pi' = (j\ 2m)\pi_{\mathfrak{m}} = \tau_r \pi_{\mathfrak{m}}$. We have $\pi_{\mathfrak{m}'} = \pi'$.*

Proof. We only modify the edges during step 2, when we remove the black vertex. The outgoing half-edges of the r-th black vertex in $\mathfrak{m}$ are labelled on the left by j and $2m$. These half-edges are part of edges connected at their other end to white vertices, because the black vertex we remove is the last. These edges are connected respectively to the half edge $\pi_{\mathfrak{m}}^{-1}(j)$ and $\pi_{\mathfrak{m}}^{-1}(2m)$.

Consider the white half-edge labelled by $\pi_{\mathfrak{m}}^{-1}(j)$. After the surgery, it is connected to the half-edge labelled $2m$. Similarly, the white half-edge labelled $\pi_{\mathfrak{m}}^{-1}(2m)$ is attached to the half-edge labelled j.

This corresponds to having $\pi_{\mathfrak{m}'}(\pi_{\mathfrak{m}}^{-1}(j)) = 2m$ and $\pi_{\mathfrak{m}'}(\pi_{\mathfrak{m}}^{-1}(2m)) = j$, and $\pi_{\mathfrak{m}'} = \pi_{\mathfrak{m}}$ for all other values. We can write this $\pi_{\mathfrak{m}'} = (j\ 2m)\pi_{\mathfrak{m}}$. □

Note that

$$\pi'^{(\epsilon)} = (\tau_r \pi_{\mathfrak{m}} \tau_r \pi_{\mathfrak{m}})|_{\epsilon^{-1}(+1)} = (\tau_r \pi_{\mathfrak{m}}^2)|_{\epsilon^{-1}(+1)} = \tau_r \pi_{\mathfrak{m}}^{(\epsilon)} = \tau_{r-1} \cdots \tau_1 .$$

The first and third equalities are Definition 3.2.7, the second one follows from the fact that $\pi_{\mathfrak{m}}(\epsilon^{-1}(+1)) = \epsilon^{-1}(-1)$ and the fact that the support of τ_r is contained in $\epsilon^{-1}(+1)$, the fourth one is a consequence of Proposition 3.3.20. This is coherent with the fact that

$$\tau_{\mathfrak{m}'} = (\tau_1, \ldots, \tau_{r-1}) . \tag{3.19}$$

Lemma 3.5.20. *Let* $\gamma' = \gamma(j\ 2m) = \gamma\tau_r$.
We have $\gamma_{\mathfrak{m}'} = \gamma'$

Proof. The white vertices are only modified when we cut the map, at step 3. The proof is similar to the one of Lemma 3.5.15. We consider the two cases of j and $2m$ in the same cycle in γ or not, and we compute $\gamma_{\mathfrak{m}} = \gamma\tau_r$. □

It follows from Lemmas 3.3.19, 3.5.19, and 3.5.20, that

$$\phi_{\mathfrak{m}'} = \tau_r\phi_{\mathfrak{m}}\tau_r\,. \tag{3.20}$$

In particular, $c(\phi_{\mathfrak{m}}) = c(\phi_{\mathfrak{m}'})$.

We can now state the counterpart of Lemma 3.5.17.

Lemma 3.5.21. *If the unitary type map* $\mathfrak{m}$ *is connected, then the map* $\mathfrak{m}'$ *has one or two connected components. Furthermore, if* j *and* $2m$ *do not belong to the same cycle in* γ*,* $\mathfrak{m}'$ *is connected.*

Proof. The proof is almost the same as for Lemma 3.5.17. Alternatively, we can prove this lemma using Proposition 3.3.23.

Let $k \geq 1$ be the number of orbits of the action of $G' := G(\gamma', \pi', \tau')$ on $[2m]$. We notice that $G := G(\gamma, \pi_{\mathfrak{m}}, \tau_{\mathfrak{m}}) = \langle \gamma', \pi', \tau_1, \dots, \tau_r\rangle$. In particular, j and $2m$ always belong to the same orbit of G. If j and $2m$ do not belong to the same orbit of G', τ_r connects two orbits of the action of G', and G has $k-1$ orbits. Conversely, if j and $2m$ belong to the same orbit of G', the actions of the two groups have the same number k of orbits.

In particular, if j and $2m$ belongs to the same cycle of γ' (or equivalently to different cycles of γ), then the two groups have the same number of orbits. We assumed that $\mathfrak{m}$ is connected so by Proposition 3.3.23, the action of G has one orbit. Thus, the action of G' has one orbit and $\mathfrak{m}'$ is connected.

In the other cases, G has k or $k-1$ orbits and necessarily, k is 1 or 2. □

Using Lemmas 3.5.21, 3.5.20 and (3.20), we can compute the genus g' of $\mathfrak{m}'$ using Euler's formula (3.13). There are three cases.

- If j and $2m$ belong to the same cycle in γ and $\mathfrak{m}'$ has two connected components, then $c(\gamma') = c(\gamma) + 1$ and $g' = g$.
- If j and $2m$ belong to the same cycle in γ and $\mathfrak{m}'$ is connected, then $c(\gamma') = c(\gamma) + 1$ and $g' = g - 1$.
- If j and $2m$ belong to two different cycles in γ, then $\mathfrak{m}'$ is connected (Lemma 3.5.21), $c(\gamma') = c(\gamma) - 1$, and $g = g'$.

Note that in these three cases, m is unchanged.

3.5.2 Proof of Theorem 3.5.1

We can now turn to the proof of Theorem 3.5.1.

Proof. Fix $\gamma = \gamma_{\boldsymbol{P}} \in \mathfrak{S}_{2m}$, $\epsilon = \epsilon_{\boldsymbol{P}}$, and $\boldsymbol{M} = \boldsymbol{M}_{\boldsymbol{P}}$. Assume first that $m = \frac{1}{2}\deg \boldsymbol{P} \geq 2$. We decompose the sum

$$\mathcal{M}_{0,l}^{(g),N}(P_1, \dots, P_lu) = (-1)^m \sum_{\substack{\mathfrak{m}\in\mathfrak{C}(g,[2m],\epsilon,\gamma)\\ \mathfrak{m}\text{ connected}}} (-1)^{c(\phi_{\mathfrak{m}})}\operatorname{tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M})$$

in two sums, each corresponding to one of the two cases of the previous construction.

We introduce the set W_{2m}^f, of monotone walks whose last step τ satisfy $\tau(2m) = 2m$, and the set W_{2m}^c of monotone walks whose last step τ satisfy $\tau(2m) \neq 2m$. The functions $\mathbb{1}_{W_{2m}^f}$ and $\mathbb{1}_{W_{2m}^c}$ are the indicator functions of those sets.

The sum corresponding to the first case, is thus by the previous surgery of Section 3.5.1

$$
\begin{aligned}
(-1)^m & \sum_{\substack{\mathfrak{m}\in\mathfrak{C}(g,m,\epsilon,\gamma)\\ \mathfrak{m}\text{ connected}}} (-1)^{c(\phi_{\mathfrak{m}})} \operatorname{tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M})\mathbb{1}_{W_{2m}^f}(\tau_{\mathfrak{m}}) \\
&= (-1)^m \sum_{\substack{\pi\in\mathfrak{S}_{2m}^{(\epsilon)}\\ \tau\in\overrightarrow{\mathcal{W}}^{r(g,m,\gamma,\pi)}(\pi^{(\epsilon)})\\ G(\gamma,\pi,\tau)\text{ is transitive}}} (-1)^{c(\gamma\pi^{-1})} \operatorname{tr}_{\gamma\pi^{-1}}(\boldsymbol{M})\mathbb{1}_{W_{2m}^f}(\tau) \\
&= (-1)^m \sum_{j\in\epsilon^{-1}(-1)} \sum_{\substack{\pi'\in\mathfrak{S}^{(\epsilon)}(I_j)\\ \pi=(j\,2m)\pi'\\ \tau\in\overrightarrow{\mathcal{W}}^{r(g,m,\gamma,\pi)}(\pi'^{(\epsilon)})\\ G(\gamma,\pi,\tau)\text{ is transitive}}} (-1)^{c(\gamma\pi^{-1})} \operatorname{tr}_{\gamma\pi^{-1}}(\boldsymbol{M}),
\end{aligned}
$$

where $r(g, m, \gamma, \pi) = c(\gamma) + c(\gamma^{-1}\pi^{-1}) - m + 2g - 2$ according to (3.13), and we used the fact that in the first case π can be rewritten $\pi'(j\,2m)$ for some $j \in \epsilon^{-1}(-1)$. Notice that the global factor $(-1)^m$ will account for a sign -1, when removing an edge and going from $2m$ white half-edges to $2m-2$ white half-edges.

We rewrite this as a sum of four terms, corresponding to the different ways of computing the genus, as explained in the last section. We interpret the new sums as series $\mathcal{M}^N_{g',l',0}(Q_1, \dots, Q_{l'})$, with l' (which corresponds to the number of vertices in the new map $\mathfrak{m}'$) and g' (the genus of the new map $\mathfrak{m}'$) two integers, and $Q_1, \dots, Q_{l'}$ monomials either in $\mathcal{X}$ or of degree 0. We introduce the notation $\boldsymbol{Q} = (Q_1, \dots, Q_{l'})$. These monomials are chosen so that the combinatorial data γ', and ϵ' described in the last section, and the tuple $\boldsymbol{M'}$ of appropriate monomials of degree 0 is such that $\gamma_{\boldsymbol{Q}} = \gamma'$, $\boldsymbol{\epsilon_q} = \epsilon'$, and $\boldsymbol{M_Q} = \boldsymbol{M'}$. The tuple $\boldsymbol{M'}$ is chosen differently depending on the sub-case, but always so that $\operatorname{tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M}) = \operatorname{tr}_{\mathfrak{m}'}(\boldsymbol{M'})$ (except for sub-case 1., see below).

There are four cases. Let us consider first the terms corresponding to sub-cases 1. and 3., which are

1. if j or $2m$ is a fixed point of $\phi_{\mathfrak{m}}$, then $c(\phi_{\mathfrak{m}'}) = c(\phi_{\mathfrak{m}}) - 1$, $l' = c(\gamma_{\mathfrak{m}'}) = c(\gamma_{\mathfrak{m}}) = l$, and $\mathfrak{m}'$ is connected. Thus, $g' = g$.

3. If j and $2m$ belong to the same cycle of $\gamma_{\mathfrak{m}}$, and $\mathfrak{m}'$ has two connected components, then $l' = c(\gamma_{\mathfrak{m}'}) = c(\gamma_{\mathfrak{m}}) + 1 = l + 1$, $c(\phi_{\mathfrak{m}'}) = c(\phi_{\mathfrak{m}})$, and $g' = g$.

In those two cases, the map $\mathfrak{m}$ is cut into two maps, with total genus equal to g. The case 1. corresponds to the degenerate case where one of the two maps has no edges, and is reduced to a vertex. We associate to it the weight $\operatorname{tr}(M_j)$ or $\operatorname{tr}(M_{2m})$.

Together, these cases account for the term

$$
-\sum_{P_l u = Q u^{-1} R u} \sum_{\substack{g_1+g_2=g\\ I\subset[l-1]}} \mathcal{M}^{(g_1),N}_{0,|I|+1}(\boldsymbol{P}|_I, Q)\mathcal{M}^{(g_2),N}_{0,|I^c|+1}(\boldsymbol{P}|_{I^c}, R).
$$

The sub-case 1. corresponds to the term for which Q or R in the sum is reduced to a monomial of degree 0, and the sub-case 3. to the other terms. When cutting the map, we obtain two connected components, each containing a vertex corresponding to part of P_l. This correspond to the fact that in the argument of the series, P_l is replaced by two monomials Q and R such that $Qu^{-1}Ru$, and one u and one u^{-1} are removed, corresponding to the two removed half-edges.

Similarly, the sub-case

2. If j and $2m$ belong to the same cycle of $\gamma_{\mathfrak{m}}$, and $\mathfrak{m}'$ is connected, then $l' = c(\gamma_{\mathfrak{m}'}) = c(\gamma_{\mathfrak{m}}) + 1 = l + 1$, $c(\phi_{\mathfrak{m}'}) = c(\phi_{\mathfrak{m}})$, and $g' = g - 1$ (by Euler formula).

corresponds to the term

$$-\sum_{P_l u = Q u^{-1} R u} \mathcal{M}^{(g-1),N}_{0,l+1}(P_1, \ldots, P_{l-1}, Q, R).$$

The sub-case

4. If j and $2m$ belong to two different cycles of $\gamma_{\mathfrak{m}}$, then $l' = c(\gamma_{\mathfrak{m}'}) = c(\gamma_{\mathfrak{m}}) - 1 = l - 1$, $c(\phi_{\mathfrak{m}'}) = c(\phi_{\mathfrak{m}})$, and $g' = g$.

corresponds to the term

$$-\sum_{i=1}^{l-1} \sum_{P_i = Q u^{-1} R} \mathcal{M}^{(g),N}_{0,l-1}(P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{l-1}, RQP_l).$$

Here, two vertices are glued together, corresponding to replacing two polynomials in the argument of the series by one: RQP_l.

We proceed similarly for the terms that correspond to the second case where $\tau(2m) \neq 2m$. The corresponding sum is

$$\begin{aligned}
&(-1)^m \sum_{\substack{\mathfrak{m} \in \mathfrak{C}(g,[2m],\epsilon,\gamma) \\ \mathfrak{m} \text{ connected}}} (-1)^{c(\phi_{\mathfrak{m}})} \operatorname{tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M}) \mathbb{1}_{W^c_{2m}}(\tau_{\mathfrak{m}}) \\
&= (-1)^m \sum_{\substack{\pi \in \mathfrak{S}^{(\epsilon)}_{2m} \\ \boldsymbol{\tau} \in \overrightarrow{\mathcal{W}}^{r(g,m,\gamma,\pi)}(\pi^{(\epsilon)}) \\ G(\gamma,\pi,\boldsymbol{\tau}) \text{ is transitive}}} (-1)^{c(\gamma\pi^{-1})} \operatorname{tr}_{\gamma\pi^{-1}}(\boldsymbol{M}) \mathbb{1}_{W^c_{2m}}(\boldsymbol{\tau}) \\
&= (-1)^m \sum_{\substack{j \in \epsilon^{-1}(+1) \\ j \neq 2m}} \sum_{\substack{\pi' \in \mathfrak{S}^{(\epsilon)}_{2m} \\ (\tau_1,\ldots,\tau_{r-1}) \in \overrightarrow{\mathcal{W}}^{r-1}(\pi'^{(\epsilon)}) \\ G(\gamma',\pi',\boldsymbol{\tau}) \text{ is transitive}}} (-1)^{c(\gamma'\pi'^{-1})} \operatorname{tr}_{\gamma'\pi'^{-1}}(\boldsymbol{M}_{(j\,2m)}),
\end{aligned}$$

where

- $\gamma' = (j\,2m)\gamma$
- $\boldsymbol{M}_{(j\,2m)} = (M_{(j\,2m)(1)}, M_{(j\,2m)(2)}, \ldots, M_{(j\,2m)(2m)})$
- $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_{r-1}, (j\,2m))$
- $r = r(g, m, \gamma, (j\,2m)\pi')$.

To go from the second to the third line, we replaced π by $\pi' = (j\,2m)\pi$.

Following the construction from last section, we get three kinds of terms corresponding to the three sub-cases from last section. The first sub-case is

1. If j and $2m$ belong to the same cycle in γ and $\mathfrak{m}'$ has two connected components, then $c(\gamma') = c(\gamma) + 1$ and $g' = g$.

It corresponds to the sum

$$\sum_{P_l u = Q u R u} \sum_{\substack{g_1 + g_2 = g \\ I \subset [2m]}} \mathcal{M}^{(g_1),N}_{0,|I|+1}(\boldsymbol{P}|_I, Qu) \mathcal{M}^{(g_2),N}_{0,|I^c|+1}(\boldsymbol{P}|_{I^c}, Ru).$$

The second sub-case is:

2. If j and $2m$ belong to the same cycle in γ and $\mathfrak{m}'$ is connected, then $c(\gamma') = c(\gamma) + 1$ and $g' = g - 1$.

corresponds to

$$\sum_{P_l u = QuRu} \mathcal{M}^{(g-1),N}_{0,l+1}(P_1,\ldots,P_{l-1},Qu,Ru).$$

Finally, the last sub-case is:

3. If j and $2m$ belong to two different cycles in γ, then $\mathfrak{m}'$ is connected (Lemma 3.5.21), $c(\gamma') = c(\gamma) - 1$, and $g = g'$.

corresponds to

$$\sum_{i=1}^{l-1} \sum_{P_i = QuR} \mathcal{M}^{(g),N}_{0,l-1}(P_1,\ldots,P_{i-1},P_{i+1},\ldots,P_{l-1},RQuP_l u).$$

Putting all the terms together, we get the induction relation of Theorem 3.5.1. □

3.5.3 Induction relation for the series $\mathcal{M}^{(g),N}_{V,l}$

We first prove that the series $\mathcal{M}^{(g),N}_{V,l}(\boldsymbol{P})$ exist with a radius of convergence that depend on g, l, and V. To that end, we show bounds on the series of maps for $V = 0$ that are a consequence of Theorem 3.5.1. Similar bounds have been obtained in the Gaussian case in [Mau06, Lemma 4.3].

Proposition 3.5.22. *Assume that for all $N \geq 1$ and all $1 \leq i \leq p$, $\|A_i^N\| \leq 1$. Let $\boldsymbol{q} = (q_1,\ldots,q_k) \in \mathcal{X}_n^k$ be monomials, and $\nu = \max_{1\leq i\leq k} \deg q_i$. We introduce the n-th Catalan number $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$.*

There exists constants $A_k > 1$, $B_k > 1$, and $C_k > 1$ that depend on k, and $D_{k,\nu} > 1$ that depends on k and ν such that for all $\boldsymbol{P} = P_1,\ldots,P_l \in \mathcal{X}_n^l$, and all $\boldsymbol{n} = (n_1,\ldots,n_k) \in \mathbf{N}^k$, we have the bound

$$\begin{aligned}\frac{1}{\boldsymbol{n}!}&|\mathcal{M}^{(g),N}_{0,\sum_i n_i + l}(\underbrace{q_1,\ldots,q_1}_{n_1 \text{ times}},\ldots,\underbrace{q_k,\ldots,q_k}_{n_k \text{ times}},P_1,\ldots,P_l)| \\ &\leq A_k^{l(2m+\nu\boldsymbol{n})} B_k^{-l} C^{g(2m+\nu\boldsymbol{n})} D^{\boldsymbol{n}} \prod_i \mathrm{Cat}_{\deg_{P_i}} \prod_{j=1}^k \mathrm{Cat}_{n_j},\end{aligned} \tag{3.21}$$

where $m = \frac{1}{2}\deg \boldsymbol{P}$.

The constants can be chosen to be

$$\begin{aligned} A_k = C_k &= \sqrt{6}\pi^{1/4} 2^{k+3} \\ B_k &= 3 \cdot 4^{k+1} \\ D_{k,\nu} &= 4k(4\mathrm{e}^{1/\mathrm{e}})^\nu. \end{aligned}$$

The proof is given in Section 3.8. The value of the constants can be improved. These bounds allow us to prove immediately that the series $\mathcal{M}^{(g),N}_{V,l}$ (see Definition 3.3.32) converge.

Corollary 3.5.23. *Let $\boldsymbol{P} = (P_1,\ldots,P_l) \in \mathcal{X}_n^l$, $\boldsymbol{q} = (q_1,\ldots,q_k) \in \mathcal{X}_n^k$ and $\boldsymbol{z} = (z_1,\ldots,z_k)$, and let $V = \sum_{i=1}^k z_i q_i$ be a potential.*

As a series in $\boldsymbol{z}$, $\mathcal{M}^{(g),N}_{V,l}(P_1,\ldots,P_l)$ converges absolutely with radius of convergence $R_{l,g,V}$ satisfying

$$R_{l,g,V} \geq \frac{1}{4A_k^{l+g} D_{k,\nu}}.$$

We can now turn to the induction relations. The induction relation from Theorem 3.5.1 translates to an induction relation on the series $\mathcal{M}^{(g),N}_{V,l}$.

Proposition 3.5.24. *Let $\boldsymbol{P} = (P_1, \dots, P_l) \in (\mathcal{X}_n)^l$, $\boldsymbol{q} = (q_1, \dots, q_k) \in (\mathcal{X}_n)^k$ and $\boldsymbol{z} = (z_1, \dots, z_k)$, and let $V = \sum_{i=1}^k z_i q_i$ be a potential. Assume that for all $1 \le i \le k$, $|z_i| < R_{l,g,V}$.*

For all $1 \le i \le n$, we have the equation

$$\begin{aligned} \sum_{\substack{g_1+g_2=g \\ I\subset[l-1]}} & \mathcal{M}^{(g_1),N}_{V,|I|+1} \otimes \mathcal{M}^{(g_2),N}_{V,|I^c|+1}(\boldsymbol{P}_I \otimes \boldsymbol{P}_{I^c} \sharp \partial P_l) + \mathcal{M}^{(g),N}_{V,l}(P_1 \otimes \cdots \otimes P_{l-1} \otimes (\mathcal{D}V)P_l) \\ &= -\mathcal{M}^{(g-1),N}_{V,l+1}(P_1 \otimes \cdots \otimes P_{l-1} \otimes \partial P_l) \\ &\quad - \sum_{j=1}^{l-1} \mathcal{M}^{(g),N}_{V,l-1}(P_1 \otimes \cdots \otimes P_{j-1} \otimes P_{j+1} \otimes \cdots \otimes P_{l-1} \otimes (\mathcal{D}P_j)P_l)\,. \end{aligned} \tag{3.22}$$

Proof. We sum on $\boldsymbol{n} \in \mathbf{N}^k$ the induction relations of Proposition 3.6.10 for

$$\mathcal{M}^{(g),N}_{0,\sum_i n_i + l}(\underbrace{q_1, \dots, q_1}_{n_1 \text{ times}}, \dots, \underbrace{q_k, \dots, q_k}_{n_k \text{ times}}, P_1, \dots, P_l),$$

times $\frac{\boldsymbol{z}^{\boldsymbol{n}}}{\boldsymbol{n}!}$. □

3.6 The multi-matrix case

Up to now, we have only considered integrals involving one Haar-distributed matrix U^N. The results obtained so far can be extended in a straightforward way to the case where we have $n \ge 1$ independent Haar-distributed matrices $U_1^N, \dots, U_n^N$. The polynomials we consider in the sequel are the non-commutative polynomials in u_i, u_i^{-1}, for $1 \le i \le n$ and a_j, a_j^* for $1 \le j \le p$. We denote this $*$-algebra by $\mathcal{A}_n$. Notice that $\mathcal{A} = \mathcal{A}_1$.

3.6.1 Weingarten calculus

As previously, we will consider a subset of monomials of $\mathcal{A}_n$, as the quantity we consider are multilinear functions which are tracial in each of their arguments. We define $\mathcal{X}_n$ the set of monomials of $\mathcal{A}_n$ of the form

$$P = M_1 u_{t_1}^{\epsilon_1} M_2 u_{t_2}^{\epsilon_2} \cdots M_d u_{t_d}^{\epsilon_d} \tag{3.23}$$

where $\epsilon\colon [d] \to \{+1, -1\}$, $t\colon [d] \to [n]$, and $\boldsymbol{M} = (M_1, \dots, M_d)$ is a d-uplet of monomials $M_j \in \mathcal{A}_n$, each of them being empty or a word in $a_1, a_1^*, \dots, a_p, a_p^*$.

We define for a tuple $\boldsymbol{P} = (P_1, \dots, P_l)$ the tuples $\boldsymbol{\epsilon_P}, \boldsymbol{t_P}, \boldsymbol{M_P}$ obtained by concatenating the tuples corresponding to each polynomial P_j, $1 \le j \le l$. We also define for $1 \le i \le n$, $\boldsymbol{\epsilon}_{\boldsymbol{P},i} = \boldsymbol{\epsilon}|_{\boldsymbol{t_P}^{-1}(i)}$, i.e. the tuple which encodes the exponents of the variables u_i only. We define the degree with respect to u_i of a monomial P, $\deg_i P$ as the number of occurrence of u_i or u_i^{-1} in P. The total degree of a tuple is $\deg_i \boldsymbol{P} = \sum_{j=1}^l \deg_i P_j$. We define $\deg P = \sum_{i=1}^n \deg_i P$, and $\deg \boldsymbol{P} = \sum_{j=1}^l \deg P_j$.

Property 3.2.9 is generalized as follows.

Definition 3.6.1. *Let $\boldsymbol{P} = (P_1, \dots, P_l) \in (\mathcal{X}_n)^l$. We define the permutation*

$$\gamma_{\boldsymbol{P}} = (1 \,\dots\, \deg P_1) \cdots (\sum_{j=1}^{l-1} \deg P_j + 1 \,\dots\, \deg \boldsymbol{P}).$$

Definition 3.6.2. *We introduce the moment with respect to the Haar measure in the multi-matrix case*

$$\alpha^N_{\boldsymbol{U},0,l}(P_1, \dots, P_l) = \mathbb{E}\left[\mathrm{Tr}(P_1) \cdots \mathrm{Tr}(P_l)\right],$$

where the expectation is under the product Haar measure $\mathrm{d}U_1^N \cdots \mathrm{d}U_n^N$.

Proposition 3.6.3. *Let $\boldsymbol{P} = (P_1, \ldots, P_l) \in (\mathcal{X}_n)^l$ and $J_i = \boldsymbol{t}_{\boldsymbol{P}}^{-1}(i)$.*
We have

$$\alpha^N_{\boldsymbol{U},0,l}(P_1, \ldots, P_l) = \sum_{\substack{\pi_1 \in \mathfrak{S}^{(\boldsymbol{\epsilon_P},1)}(J_1) \\ \pi_2 \in \mathfrak{S}^{(\boldsymbol{\epsilon_P},2)}(J_2) \\ \cdots \\ \pi_n \in \mathfrak{S}^{(\boldsymbol{\epsilon_P},n)}(J_n)}} \left(\prod_{i=1}^{k} \mathrm{Wg}_N(\pi_i^{(\epsilon_{\boldsymbol{P}}^{(i)})}) \right) \mathrm{Tr}_{\gamma_{\boldsymbol{P}} \pi_1^{-1} \cdots \pi_n^{-1}}(\boldsymbol{M_P}). \tag{3.24}$$

This Proposition is obtained by applying the following Lemma n times.

Lemma 3.6.4. *Let $\boldsymbol{M_P} = (M_1, \ldots, M_{\deg \boldsymbol{P}})$ Let $\tilde{\boldsymbol{M}} = (\tilde{M}_i, 1 \le i \le \deg \boldsymbol{P})$, defined by $\tilde{M}_i = M_i$ if $t_i = 1$, and $\tilde{M}_i = M_i u_{t_i}^{\epsilon_{\boldsymbol{P}}(i)}$ otherwise. Then, we have the expectation with respect to U_1^N only*

$$\mathbb{E}_{U_1^N}[\mathrm{Tr}(P_1) \cdots \mathrm{Tr}(P_l)] = \sum_{\pi_1 \in \mathfrak{S}^{(\epsilon_{\boldsymbol{P}},1)}(J_1)} \mathrm{Wg}_N(\pi_1^{(\epsilon_{\boldsymbol{P}},1)}) \, \mathrm{Tr}_{\gamma_{\boldsymbol{P}} \pi_1^{-1}}(\tilde{\boldsymbol{M}}).$$

Proof of Lemma 3.6.4. Let $I \subset [l]$ be the subset of indices i such that P_i contains a letter u_1 or u_1^*. Denote by c_i the cycle in γ that corresponds to P_i, i.e.

$$c_i = (\sum_{j=1}^{i-1} \deg P_j + 1 \cdots \sum_{j=1}^{i-1} \deg P_j + \deg P_i) \, .$$

We have

$$\mathbb{E}_{U_1^N}[\mathrm{Tr}(P_1) \cdots \mathrm{Tr}(P_l)] = \left(\prod_{i \notin I} \mathrm{Tr}_{c_i}(P_i) \right) \mathbb{E} \left[\prod_{i \in I} \mathrm{Tr}(P_i) \right] . \tag{3.25}$$

Furthermore, if we set $S = \bigcup_{i \notin I} \mathrm{Supp}\, c_i$ and $\gamma'' = (\prod_i c_i)\,|_S$, we can rewrite the terms

$$\left(\prod_{i \notin I} \mathrm{Tr}_{c_i}(P_i) \right) = \mathrm{Tr}_{\gamma''}(\tilde{\boldsymbol{M}}|_S) \, .$$

Considering only the second term in the right side of (3.25) and using the cyclic property of the trace, we can assume that the last factor of each polynomial P_i is a u_1 or a u_1^*. Let $J_1 = t^{-1}(1) = \{p_1 < p_2 < \ldots < p_q\}$. We let $\gamma' = \mathrm{Tr}(\gamma_{\boldsymbol{P}}; J_1)$. Proposition 3.2.9, shows that

$$\mathbb{E}_{U_1^N}[\mathrm{Tr}(P_1) \cdots \mathrm{Tr}(P_l)] = \sum_{\pi_1 \in \mathfrak{S}^{(\epsilon_{\boldsymbol{P}},1)}(J_1)} \mathrm{Wg}_N(\pi_1^{(\epsilon_{\boldsymbol{P}},1)}) \, \mathrm{Tr}_{\gamma' \pi_1^{-1}}(\boldsymbol{M}'),$$

where $\boldsymbol{M}' = (M_i', i \in J_1)$ is defined by

$$M_i' = M_{p_{i-1}+1} u_{t(p_{i-1}+1)}^{\epsilon_{\boldsymbol{P}}(p_{i-1}+1)} M_{p_{i-1}+2} u_{t(p_{i-1}+2)}^{\epsilon_{\boldsymbol{P}}(p_{i-1}+2)} \cdots M_{p_i - 1} u_{t(p_i-1)}^{\epsilon_{\boldsymbol{P}}(p_i - 1)} M_{p_i},$$

with the convention $p_0 = 0$. This is equal to

$$\mathbb{E}_{U_1^N}[\mathrm{Tr}(P_1) \cdots \mathrm{Tr}(P_l)] = \sum_{\pi_1 \in \mathfrak{S}^{(\epsilon_{\boldsymbol{P}},1)}(J_1)} \mathrm{Wg}_N(\pi_1^{(\epsilon_{\boldsymbol{P}},1)}) \, \mathrm{Tr}_{\gamma_{\boldsymbol{P}} \pi_1^{-1}}(\tilde{\boldsymbol{M}}).$$

□

3.6.2 Multicolored maps of unitary type

We now generalize the notion of a map of unitary type to address the multi-matrix case.

Definition 3.6.5. *Let I be a finite subset of $\mathbf{N}^*$. A* ***multicolored map of unitary type*** *with n colors, with labels in I, and with r_i vertices of color i for $1 \leq i \leq n$, is an oriented map with vertices colored in white or in one of n colors, and colored half-edges which can be of any of the n colors such that*

- *there are r_i vertices of color i for $1 \leq i \leq n$, which are alternated of degree 4 and numbered from 1 to r_i;*
- *the half-edges connected to a vertex of color i (which is not white) are of color i as well;*
- *there are $|I|$ half-edges that are connected to white vertices. Each element of I labels exactly one of these half-edges;*
- *each half-edges connected to a white vertex is colored in one of the n colors;*
- *each edge is composed of two half-edges of the same color;*
- *if an oriented edge connects the vertex of color i numbered l_1 to the vertex of color i numbered l_2 then $l_1 < l_2$.*

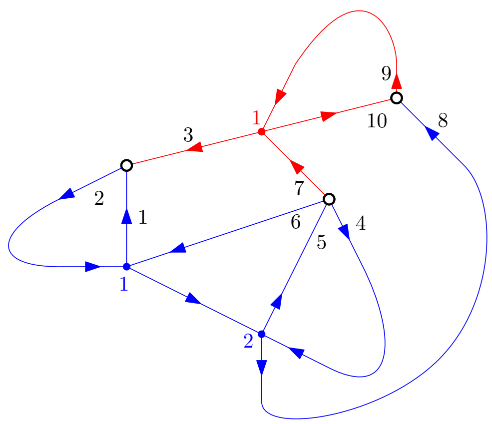


Figure 3.14: A muticolored map of unitary type with two colors. The integers $3, 7, 10, 9$ label red half-edges, and the other elements of $[10]$ label blue half-edges.

Remark 3.6.6. Notice that we can erase the colors of a multicolored unitary map to obtain a (monocolored) map of unitary type. To do so, we proceed as follows:

- each colored half-edge connected to a white vertex becomes a white half-edge;
- each colored half-edge connected to a colored vertex becomes a black half-edge;
- each colored vertex becomes a black vertex.

The resulting map is a map of unitary type with $\sum_i r_i$ black vertices and labels in I.

As in Section 3.3.2, we define for a multicolored map $\mathfrak{m}$ of unitary type, with n colors, the following permutations. We construct permutations $\gamma_{\mathfrak{m}}$, $\pi_{\mathfrak{m}}$, and $\phi_{\mathfrak{m}}$, and the tuple $\boldsymbol{\epsilon}_{\mathfrak{m}}$, as for a monocolored map of unitary type. If the i-th labelled half-edge is of color j, we set $\boldsymbol{t}_{\mathfrak{m}}(i) = j$. We then define $J_{\mathfrak{m},i} = \boldsymbol{t}_{\mathfrak{m}}^{-1}(i)$. We set $\boldsymbol{\epsilon}_{\mathfrak{m},i} = \boldsymbol{\epsilon}_{\mathfrak{m}}|_{J_{\mathfrak{m},i}}$ for all $1 \leq i \leq k$.

For each color $i \in [n]$, we consider the edges of this color. We then define as previously a permutation $\pi_{\mathfrak{m},i} \in \mathfrak{S}^{(\boldsymbol{\epsilon}_{\mathfrak{m},i})}(J_{\mathfrak{m},i})$ describing these edges and the vertices of color i. Finally, if we consider the vertices of color i, we can associate to the j-th vertex of color i the transposition $\tau_{i,j}$ as previously. We set $\tau_{\mathfrak{m}} = (\tau_{i,j}, i \in [n], j \in [r_i])$. Notice that by construction, we have $\pi_{\mathfrak{m}} = \pi_{\mathfrak{m},1}\pi_{\mathfrak{m},2}\cdots\pi_{\mathfrak{m},n}$.

Definition 3.6.7. *Let $n \geq 1$ and $g \geq 0$ be integers, and I be a finite subset of the positive integers. Let $\boldsymbol{r} = (r_1, \dots, r_n) \in \mathbf{N}^n$, $\gamma \in \mathfrak{S}(I)$, $\boldsymbol{t}\colon I \to [n]$ and $\boldsymbol{\epsilon}\colon I \to \{\pm 1\}$.*

We denote by $\mathfrak{C}^{\boldsymbol{r}}(I, \boldsymbol{\epsilon}, \gamma, \boldsymbol{t})$ the set of nondecreasing multicolored maps of unitary type $\mathfrak{m}$ with n colors, with label set I, and with r_i vertices of color i, for $1 \leq i \leq n$, such that $\gamma_{\mathfrak{m}} = \gamma$, $\boldsymbol{\epsilon}_{\mathfrak{m}} = \boldsymbol{\epsilon}$, and $\boldsymbol{t}_{\mathfrak{m}} = \boldsymbol{t}$.

We denote by $\mathfrak{C}(g, I, \boldsymbol{\epsilon}, \gamma, \boldsymbol{t})$ the set of nondecreasing multicolored maps of unitary type $\mathfrak{m}$ with n colors, with label set I, and of genus g, such that $\gamma_{\mathfrak{m}} = \gamma$, $\boldsymbol{\epsilon}_{\mathfrak{m}} = \boldsymbol{\epsilon}$, and $\boldsymbol{t}_{\mathfrak{m}} = \boldsymbol{t}$.

We then have the analog of Theorem 3.3.22

Theorem 3.6.8. *Let $n \geq 1$ be an integer, and I be a finite subset of the positive integers. Let $\boldsymbol{r} = (r_1, \dots, r_n) \in \mathbf{N}^n$, $\gamma \in \mathfrak{S}(I)$, $\boldsymbol{t}\colon I \to [n]$ and $\boldsymbol{\epsilon}\colon I \to \{\pm 1\}$. Define $J_i = \boldsymbol{t}^{-1}(i)$ and $\boldsymbol{\epsilon}_i = \boldsymbol{\epsilon}|_{J_i}$ for $1 \leq i \leq n$.*

The previous construction gives a bijection between $\mathfrak{C}^{\boldsymbol{r}}(I, \boldsymbol{\epsilon}, \gamma, \boldsymbol{t})$ and

$$\bigcup_{\pi_1 \in \mathfrak{S}^{(\epsilon_1)}(J_1), \dots, \pi_n \in \mathfrak{S}^{(\epsilon_n)}(J_n)} \prod_{i=1}^{n} \{\pi_i\} \times \overrightarrow{\mathcal{W}}^{r_i}(\pi_i^{(\epsilon_i)}).$$

Proof. The proof is very similar to the one of Theorem 3.3.22. By considering each color, we prove that the construction does give a map $\mathfrak{C}^{\boldsymbol{r}}(I, \boldsymbol{\epsilon}, \gamma, \boldsymbol{t}) \to \bigcup_{\pi_1 \in \mathfrak{S}^{(\epsilon_1)}(J_1), \dots, \pi_n \in \mathfrak{S}^{(\epsilon_n)}(J_n)} \prod_{i=1}^{n} \{\pi_i\} \times \overrightarrow{\mathcal{W}}^{r_i}(\pi_i^{(\epsilon_i)})$.

We can construct its inverse exactly as in the proof for the case with one unitary matrix, by constructing the edges for the color 1, then for the color 2, etc. More precisely, we first consider the color 1. At this step, we leave untouched the half-edges of color $2, \dots, n$. We construct the incidence relation for the edges of color 1 using the data of $(\tau_{1,j})_{j \in [r_1]}$ and π_1. When this is finished, all the half-edges of color 1 are part of some edge.

We then turn to the half-edges of color 2. Using the data of $(\tau_{2,j})_{j \in [r_2]}$ and π_2, we construct the incidence relation for the edges of color 2. We do this until we reach color n and obtain a multicolored map of unitary type. □

It follows directly by erasing the colors (see Remark 3.6.6) and Proposition 3.3.23 that we have the following proposition.

Proposition 3.6.9. *Let $\mathfrak{m}$ be a multicolored map of unitary type with n colors. The map $\mathfrak{m}$ is connected if and only if the group $\langle \gamma_{\mathfrak{m}}, \pi_{\mathfrak{m},1}, \dots, \pi_{\mathfrak{m},n}, \tau_{i,j}, 1 \leq i \leq n, 1 \leq j \leq r_i \rangle$ is transitive, with $\tau_{\mathfrak{m}} = (\tau_{i,j}, i \in [n], j \in [r_i])$.*

Using Proposition 3.2.12, 3.6.3, and 3.6.8, we can compute the moments with no potential (for $N \geq m = \frac{1}{2} \deg \boldsymbol{P}$). Let $\boldsymbol{P} = (P_1, \dots, P_l) \in (\mathcal{X}_n)^l$, we have

$$\begin{aligned} \alpha^N_{\boldsymbol{U},0,l}(P_1, \dots, P_l) &= N^{-m} \sum_{\substack{\pi_1 \in \mathfrak{S}^{(\epsilon_{\boldsymbol{P}},1)}(J_1) \\ \pi_2 \in \mathfrak{S}^{(\epsilon_{\boldsymbol{P}},2)}(J_2) \\ \dots \\ \pi_n \in \mathfrak{S}^{(\epsilon_{\boldsymbol{P}},n)}(J_n)}} \sum_{r_1, \dots, r_n \geq 0} \left(\frac{-1}{N}\right)^{\boldsymbol{r}} \prod_{i=1}^{n} \overrightarrow{w}^{r_i}(\pi_i^{(\epsilon_i)}) \operatorname{Tr}_{\gamma_{\boldsymbol{P}} \pi_1^{-1} \cdots \pi_n^{-1}}(\boldsymbol{M}_{\boldsymbol{P}}) \\ &= N^{-m} \sum_{\boldsymbol{r} \in \mathbf{N}^k} \sum_{\mathfrak{m} \in \mathfrak{C}(\boldsymbol{r}_{[2m], \boldsymbol{\epsilon}_{\boldsymbol{P}}, \gamma_{\boldsymbol{P}}, \boldsymbol{t}})} \left(\frac{-1}{N}\right)^{\boldsymbol{r}} \operatorname{Tr}_{\gamma \pi_1^{-1} \cdots \pi_n^{-1}}(\boldsymbol{M}_{\boldsymbol{P}}), \end{aligned} \tag{3.26}$$

where we use the notation $x^{\boldsymbol{r}} = x^{\sum_i r_i}$, for any $x \in \mathbb{R}$.

We then compute the cumulants for no potential, when $N \geq m$,

$$\mathcal{W}^N_{\boldsymbol{U}^{\boldsymbol{N}},0,l}(P_1, \dots, P_l) = N^{-m} \sum_{\boldsymbol{r} \in \mathbf{N}^n} \sum_{\substack{\mathfrak{m} \in \mathfrak{C}^{\boldsymbol{r}}([2m], \boldsymbol{\epsilon}_{\boldsymbol{P}}, \gamma_{\boldsymbol{P}}, \boldsymbol{t}) \\ \mathfrak{m} \text{ connected}}} \left(\frac{-1}{N}\right)^{\boldsymbol{r}} \operatorname{Tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M}_{\boldsymbol{P}}). \tag{3.27}$$

We now rewrite this sum using the genus of the maps rather than the number of colored vertices. In this context, the Euler formula becomes

$$2 - 2g_{\mathfrak{m}} = c(\gamma_{\mathfrak{m}}) + c(\phi_{\mathfrak{m}}) - m - \sum_{i=1}^{k} r_i.$$

We thus get the renormalized cumulant $\tilde{\mathcal{W}}^N_{\boldsymbol{U^N},0,l}$

$$\begin{aligned}\tilde{\mathcal{W}}^N_{\boldsymbol{U^N},0,l}(P_1,\dots,P_l) &= N^{l-2}\mathcal{W}^N_{\boldsymbol{U^N},0,l}(P_1,\dots,P_l)\\ &= (-1)^{m+l}\sum_{g\geq 0}\frac{1}{N^{2g}}\sum_{\substack{\mathfrak{m}\in\mathfrak{C}(g,[2m],\boldsymbol{\epsilon_P},\gamma_{\boldsymbol{P}},\boldsymbol{t})\\ \mathfrak{m}\text{ connected}}}(-1)^{c(\phi_{\mathfrak{m}})}\operatorname{tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M_P}).\end{aligned} \tag{3.28}$$

Let us recall the relevant notation. Here l is the number of monomials or the number of white vertices. In particular, we have $c(\gamma_{\boldsymbol{P}}) = l$. The set of maps $\mathfrak{C}(g,[2m],\boldsymbol{\epsilon_P},\gamma_{\boldsymbol{P}},\boldsymbol{t})$ was introduced in Definition 3.6.7. As in Section 3.3.4, the coefficients in the $1/N^2$ expansion are sums of maps, with a weight determined by the permutation of the faces $\phi_{\mathfrak{m}}$ and the tuple of matrices $\boldsymbol{M}$.

The term of order $2g$ is then

$$\mathcal{M}^{(g),N}_{0,l}(P_1,\dots,P_l) = (-1)^{m+l}\sum_{\substack{\mathfrak{m}\in\mathfrak{C}(g,[2m],\boldsymbol{\epsilon_P},\gamma_{\boldsymbol{P}},\boldsymbol{t})\\ \mathfrak{m}\text{ connected}}}(-1)^{c(\phi_{\mathfrak{m}})}\operatorname{tr}_{\phi_{\mathfrak{m}}}(\boldsymbol{M_P}).$$

We then define the formal cumulant as

$$\mathcal{M}^{(g),N}_{V,l}(P_1,\dots,P_l) = \sum_{\boldsymbol{n}\in\mathbf{N}^k}\frac{\boldsymbol{z^n}}{\boldsymbol{n}!}\mathcal{M}^{(g),N}_{0,l}(\underbrace{q_1,\dots,q_1}_{n_1\text{ times}},\dots,\underbrace{q_k,\dots,q_k}_{n_k\text{ times}},P_1,\dots,P_l),$$

as previously.

3.6.3 Induction relation

We will now deduce from the relations obtained in Section 3.5 similar relations in the multi-matrix case.

Proposition 3.6.10. *Let* $\boldsymbol{P} = (P_1,\dots,P_l) \in (\mathcal{X}_n)^l$, $i\in[n]$ *and* $g\geq 1$.

If $\frac{1}{2}\deg_i \boldsymbol{P} \geq 2$, *then we have the equation*

$$\begin{aligned}&\sum_{\substack{g_1+g_2=g\\ I\subset[l-1]}}\mathcal{M}^{(g_1),N}_{V,|I|+1}\otimes\mathcal{M}^{(g_2),N}_{V,|I^c|+1}(\boldsymbol{P}_I\otimes\boldsymbol{P}_{I^c}\#\partial_i P_l) + \mathcal{M}^{(g),N}_{V,l}(P_1\otimes\cdots\otimes P_{l-1}\otimes(\mathcal{D}_iV)P_l)\\ &\quad= -\mathcal{M}^{(g-1),N}_{V,l+1}(P_1\otimes\cdots\otimes P_{l-1}\otimes\partial_i P_l)\\ &\qquad -\sum_{j=1}^{l-1}\mathcal{M}^{(g),N}_{V,l-1}(P_1\otimes\cdots\otimes P_{j-1}\otimes P_{j+1}\otimes\cdots\otimes P_{l-1}\otimes(\mathcal{D}_iP_j)P_l).\end{aligned} \tag{3.29}$$

Here ∂_i *and* $\mathcal{D}_i$ *are the non-commutative and cyclic derivative with respect to* u_i, *for* $i\in[n]$.

This Proposition is proved as in Section 3.5. If no polynomial of $\boldsymbol{P}$ contains a u_i then the equation is trivial. Thus, we can assume by symmetry that $\deg_i P_l \geq 1$. We cut the maps from the sum $\mathcal{W}^{(g),N}_{0,l}(\boldsymbol{P})$ as in Section 3.5.1. Notice that in this construction, we only modify edges of the color i so we can use the exact same arguments. We thus obtain the wanted equation.

3.7 The Dyson-Schwinger equation and the topological expansion

We now work in the multi-matrix setting. All the maps involved will be multicolored maps. The induction equations obtained in Section 3.5 are related to the Dyson-Schwinger equations for unitary matrices. In this section, we introduce the Dyson-Schwinger lattice of equations for the renormalized cumulants $\tilde{\mathcal{W}}_{V,l}^N = N^{l-2}\mathcal{W}_{V,l}^N$. Together with the induction relations derived in Section 3.5, they allow us to show that the renormalized cumulants $\tilde{\mathcal{W}}_{V,l}^N$ admit an asymptotic topological expansion as $N \to \infty$. The methods used in this section are heavily inspired from [GN15].

3.7.1 Scalar product and parametric norms on $\mathcal{A}_n$

Following [GN15], we introduce some useful notions about non-commutative polynomials. The vector space $\mathcal{A}$ – the algebra of non-commutative polynomials – admits a countable basis, which is the set $\hat{\mathcal{X}}_n$ of all words in the letters

$$u_1, u_1^*, \ldots, u_n, u_n^*, a_1, a_1^*, \ldots, a_p, a_p^*,$$

plus the empty word 1. Notice that this set contains $\mathcal{X}_n$, the set of such words that finishes by a u_i or a u_i^*. Let $\langle \cdot, \cdot \rangle$ be the scalar product that makes this basis orthonormal. In particular, $\mathcal{B}^\perp$ is the algebra generated by the polynomials with no constant term, i.e. without factors u_i or u_i^*.

Definition 3.7.1. *Let $\xi \geq 1$. The ξ-norm is $\|\cdot\|_\xi$ defined by*

$$\|P\|_\xi = \sum_{Q \in \hat{\mathcal{X}}} |\langle P, Q \rangle| \xi^{\deg Q},$$

for $P \in \mathcal{A}$. We write $\mathcal{B}_\xi^\perp$ the completion of the algebra $\mathcal{B}^\perp$ in the ξ-norm $\|\cdot\|_\xi$.

This norm is a deformation of the ℓ^1 norm that takes into account the degree of the basis monomials. The usual ℓ^1 norm will in many case not be the appropriate norm, as the effect of many operators we consider in the sequel depends on the degree of the monomial it is applied to.

Example 3.7.2. This norm is deformation of the ℓ^1 norm, recovered when considering the 1-norm. For instance the 1-norm of the potential we consider is

$$\|V\|_1 = \sum_{i=1}^{k} |z_i|.$$

This notion of norm allows us to define the parametric ξ-norm of a linear operator or form.

Definition 3.7.3. *Let T be an operator on $\mathcal{A}$ and $\xi, \xi' \geq 1$. Its (ξ, ξ')-norm is*

$$\|T\|_{\xi,\xi'} = \sup_{P \in \mathcal{A}} \frac{\|TP\|_{\xi'}}{\|P\|_\xi}.$$

When $\xi = \xi'$ we write $\|T\|_\xi = \|T\|_{\xi,\xi}$.

Similarly, let $\tau\colon \mathcal{A} \to \mathbb{C}$ be a linear form. Its ξ-norm is

$$\|\tau\|_\xi = \sup_{P \in \mathcal{A}} \frac{|\tau(P)|}{\|P\|_\xi}.$$

A particularly important sort of linear forms are tracial states.

Definition 3.7.4. *Let $\mathcal{C}$ be a unital $*$-algebra. A **tracial state** on $\mathcal{C}$ is a linear form $\tau\colon \mathcal{C} \to \mathbb{C}$ such that for any $P, Q \in \mathcal{C}$, we have*

- *$\tau(\mathbf{1}) = 1$, where $\mathbf{1}$ is the empty word;*

- $\tau(PQ) = \tau(QP)$;
- $\tau(PP^*) \geq 0$.

Remark 3.7.5. The normalized trace tr is a tracial state on $\mathcal{B}$. Under Hypothesis 3.1.3, the Cauchy-Schwarz inequality implies that $\|\operatorname{tr}\|_1 \leq 1$. Furthermore, we have $\operatorname{tr}(\mathbf{1}) = \operatorname{tr}(\mathrm{Id}) = 1$ where Id is the identity matrix, and thus

$$\|\operatorname{tr}\|_1 = 1.$$

Assuming furthermore Hypothesis 3.1.1, we have that $\tilde{\mathcal{W}}^N_{V,1}$ is a tracial state on $\mathcal{A}_n$, with $\|\tilde{\mathcal{W}}^N_{V,1}\|_1 = 1$.

3.7.2 The Dyson-Schwinger equations for the unitary matrices

Let σ be a tracial state on $\mathcal{B}$. A tracial state μ on $\mathcal{A}$ is a solution to the Dyson-Schwinger problem with initial value σ if for all $P \in \mathcal{A}$,

$$\begin{cases} \mu \otimes \mu(\partial_i P) + \mu(\mathcal{D}_i V \cdot P) & = 0, \text{ for } 1 \leq i \leq n \\ \mu|_{\mathcal{B}} & = \sigma\,, \end{cases} \tag{3.30}$$

where ∂_i and $\mathcal{D}_i$ are the non-commutative derivative and cyclic derivative with respect to u_i, see Definitions 3.5.2 and 3.5.5.

It has been shown in [CGM09] that there exists a solution to this problem when $\operatorname{Tr} V = \operatorname{Tr} V^*$ (which implies that $\operatorname{Tr} V$ is real), and that the solution is unique for a potential V small enough (i.e. $\sum_{i=1}^k |z_i| < \epsilon$ for some $\epsilon > 0$). Notice that for all $N \geq 1$, $\mathcal{M}^{(0),N}_{V,1}$ is a solution to (3.30) with $\sigma = \operatorname{tr}_N$. In [GN15], a family of equations that generalize (3.30) was studied. The renormalized cumulants $\tilde{\mathcal{W}}^N_{V,l}$ are solution to these equations. We reproduce them here.

Proposition 3.7.6 ([GN15, Proposition 20])**.** *Assume Hypothesis 3.1.1. The renormalized cumulants* $\{\tilde{\mathcal{W}}^N_{V,l}\}_{l\geq 1}$ *satisfy the equation*

$$\begin{aligned} \sum_{I\subset[l-1]} \tilde{\mathcal{W}}^N_{V,|I|+1} \otimes \tilde{\mathcal{W}}^N_{V,|I^c|+1}(\boldsymbol{P}_I \otimes \boldsymbol{P}_{I^c} \# \partial_i P_l) + \tilde{\mathcal{W}}^N_{V,l}\left(\boldsymbol{P}_{[l-1]} \otimes (\mathcal{D}_i V \cdot P_l)\right) \\ = -\sum_{j=1}^{l-1} \tilde{\mathcal{W}}^N_{V,l-1}(P_1 \otimes \cdots \otimes \check{P}_j \otimes \cdots \otimes P_{l-1} \otimes (\mathcal{D}_i P_j \cdot P_l)) \\ - \frac{1}{N^2}\tilde{\mathcal{W}}^N_{V,l+1}(\boldsymbol{P}_{[l-1]} \otimes \partial_i P_l), \end{aligned} \tag{3.31}$$

where $\check{P}_j$ means that the factor P_j is omitted.

The series of maps $\mathcal{M}^{(g),N}_{V,l}$ satisfy similar equations (see (3.22)).

3.7.3 Radius of convergence of the series $\mathcal{M}^{(g),N}_{V,l}$

Before giving the proof of Theorem 3.1.4, we show that all the terms $\mathcal{M}^{(g),N}_{V,l}$ have a radius of convergence greater than some $R_V > 0$. We can apply the gradient trick from [GN15] that we explain in Section 3.9 to the equations from Proposition 3.6.10. To do so, we introduce some notation, motivated in Section 3.9.

Definition 3.7.7. *Let $P \in \hat{\mathcal{X}}_n$ be a monomial, we define*

$$\begin{aligned} \Delta_i P = &\sum_{P=P_1u_iP_2}\left(\sum_{P_2P_1=Q_1u_iQ_2} Q_1u_i \otimes Q_2u_i - \sum_{P_2P_1=Q_1u_i^{-1}Q_2} Q_1 \otimes Q_2\right) \\ &- \sum_{P=P_1u_i^{-1}P_2}\left(\sum_{P_2P_1=Q_1u_iQ_2} Q_1 \otimes Q_2 - \sum_{P_2P_1=Q_1u_i^{-1}Q_2} u_i^{-1}Q_1 \otimes u_i^{-1}Q_2\right). \end{aligned}$$

*The **reduced Laplacian** $\Delta\colon \mathcal{A}_n \to \mathcal{A}_n^{\otimes 2}$ is then defined by*

$$\Delta = \sum_{i=1}^{n} \Delta_i \,.$$

Let $(P,Q) \in \mathcal{A}_n^2$, we define the operator $\mathcal{P}^Q$ by

$$\mathcal{P}^Q P = \sum_{i=1}^{n} (\mathcal{D}_i Q)(\mathcal{D}_i P) \,.$$

We define the operator D_i, which acts on a monomial P by $D_i P = \deg_i(P)P$, and D by

$$D = \sum_{i=1}^{n} D_i \,.$$

Furthermore, for an operator T, we introduce its regularization $\bar{T} = TD^{-1}$.

Definition 3.7.8 ([GN15, Definition 13])**.** *Let Π be the orthogonal projection of the polynomials onto $\mathcal{B}^{\perp}$, the algebra of polynomials without a degree 0 term. Let $\Pi' = \mathrm{Id} - \Pi$ be the complementary projection of the polynomials onto $\mathcal{B}$.*

Let τ be a tracial state. We define

$$T_\tau = (\mathrm{Id} \otimes \tau + \tau \otimes \mathrm{Id})\Delta \,.$$

*The **master operator** is*

$$\Xi_\tau^V = \mathrm{Id} + \Pi \bar{T}_\tau + \bar{\mathcal{P}}^V \,.$$

Where $\bar{T}_\tau$ and $\bar{\mathcal{P}}^V$ are the regularization (see Definition 3.7.7) of T_τ and $\mathcal{P}^V$.

Applying the gradient trick, we obtain for $(g,l) \neq (0,1)$,

$$\begin{aligned}
\mathcal{M}_{V,l}^{(g),N}(\boldsymbol{P}_{[l-1]} \otimes \Xi^V_{\mathcal{M}_{V,1}^{(0),N}} P_l) = &-\mathcal{M}_{V,l+1}^{(g-1),N}(\boldsymbol{P}_{[l-1]} \otimes \bar{\Delta} P_l) \\
&- \sum_{h=1}^{g-1} \mathcal{M}_{V,l}^{(g-h),N}(\boldsymbol{P}_{[l-1]} \otimes \mathcal{M}_{V,1}^{(h),N} \otimes \mathrm{Id})(\bar{\Delta} P_l) \\
&- \sum_{\substack{\emptyset \subsetneq I \subsetneq [l-1] \\ g_1+g_2=g}} \mathcal{M}_{V,|I|+1}^{(g_1),N} \otimes \mathcal{M}_{V,|I^c|+1}^{(g_2),N}(\boldsymbol{P}_I \otimes \boldsymbol{P}_{I^c} \# \bar{\Delta} P_l) \\
&- \sum_{j=1}^{l-1} \mathcal{M}_{V,l-1}^{(g),N}(P_1, \ldots, \check{P}_j, \ldots, P_{l-1}, \bar{\mathcal{P}}^{P_j} P_l).
\end{aligned} \tag{3.32}$$

Proposition 3.7.9. *Fix a potential $V = \sum_{i=1}^{k} z_i q_i$, with $q_1, \ldots, q_k \in \mathcal{X}_n$. Let $g \geq 0$, $l \geq 1$, $\boldsymbol{P} \in \mathcal{A}_n^l$.*

The radius of convergence of $\mathcal{M}_{V,l}^{(g),N}(\boldsymbol{P})$ depends only on k and $q_1, \ldots, q_k$, and is greater than $R_V = \min(\frac{1}{2}(4A_k D_{k,\nu})^{-1}, \frac{1}{2k\nu(4A_k + \frac{2k+2}{B_k})^\nu})$, where $\nu = \max_{1 \leq i \leq k} \deg q_i$ and the constants A_k, B_k, and $D_{k,\nu}$ are those of Proposition 3.5.22.

Proof. Let $\boldsymbol{P} \in (\mathcal{X}_n)^l$ be monomials. As $\mathcal{M}_{V,l}^{(g),N}$ is linear in each polynomial P_i, the result follows from the case where the P_i are monomials. Using Proposition 3.5.22, the series $\mathcal{M}_{V,1}^{(0),N}(\boldsymbol{P})$ can be bounded as follows

$$|\mathcal{M}_{V,1}^{(0),N}(\boldsymbol{P})| \leq \sum_{\boldsymbol{n}\in\mathbf{N}^k} \frac{\boldsymbol{z}^{\boldsymbol{n}}}{\boldsymbol{n}!} |\mathcal{M}^{(0),N}_{0,\sum_i n_i+l}(\underbrace{q_1,\ldots,q_1}_{n_1 \text{ times}},\ldots,\underbrace{q_k,\ldots,q_k}_{n_k \text{ times}},P_1,\ldots,P_l)|$$
$$\leq \frac{(4A_k)^{\deg \boldsymbol{P}}}{B_k} \sum_{\boldsymbol{n}\in\mathbf{N}^k} \boldsymbol{z}^{\boldsymbol{n}} (4A_k D_{k,\nu})^{\boldsymbol{n}}$$
$$\leq \frac{(4A_k)^{\deg \boldsymbol{P}}}{B_k} \prod_{i=1}^{k} \frac{1}{1-4A_kD_{k,\nu}z_i},$$

where Proposition 3.5.22 is used on the second line. Notice that the radius of convergence of the series does not depend on $\boldsymbol{P}$.

Assuming that $\|\boldsymbol{z}\|_\infty < \frac{1}{2}(4A_kD_{k,\nu})^{-1}$, we get

$$\|\mathcal{M}_{V,1}^{(0),N}\|_{4A_k} \leq \frac{2^k}{B_k}.$$

Define

$$K(\xi,V) = \frac{2^{k+1}}{B_k}\frac{4A_k}{\xi-4A_k} + \|\Pi V\|_1 \nu \xi^{\nu},$$

where as before $\nu = \max_{1\leq i\leq k} \deg q_i$. Choose $\xi = 4A_k + \frac{2^{k+2}}{B_k}$. Then, assuming than

$$\|\Pi V\|_1 = \sum_{i=1}^{k} |z_i| < \frac{1}{2\nu\xi^{\nu}},$$

where Π is the projection operator introduced in Definition 3.7.8, we have $K(\xi,V)<1$ and $\Xi^V_{\mathcal{M}_{V,1}^{(0),N}}$ is an invertible operator $\mathcal{B}_\xi^\perp \to \mathcal{B}_\xi^\perp$. Note that this is satisfied if $\|\boldsymbol{z}\|_\infty < \frac{1}{2k\nu\xi^\nu}$. We thus set $R_V = \min(\frac{1}{2}(4A_kD_{k,\nu})^{-1}, \frac{1}{2k\nu\xi^\nu})$.

We then proceed by induction. Assume that for all $(g',l') < (g,l)$ (with the lexicographic order), and for all $\boldsymbol{P} \in \mathcal{X}_n^{l'}$, the series $\mathcal{M}_{V,l'}^{(g'),N}(\boldsymbol{P})$ has a radius of convergence greater than R_V. Then, the right side of (3.32) is a holomorphic function that is defined on a polydisc of radius R_V. The left side is a holomorphic function defined on a polydisc of radius $R_{l,g,V}$ which coincide with the right side. Thus, it can be extended to a holomorphic function on a polydisc of radius R_V. The fact that $\Xi^V_{\mathcal{M}_{V,1}^{(0),N}}$ is invertible allows us to conclude. □

3.7.4 The topological expansion: proof of Theorem 3.1.4

We introduce the truncated formal cumulant (cf. Definition 3.3.32)

$$S_{V,l}^{(g),N} = \sum_{h=0}^{g} \frac{1}{N^{2h}} \mathcal{M}_{V,l}^{(h),N}.$$

We will show that the cumulants $\tilde{\mathcal{W}}_{V,l}^N$ admit a topological expansion by bounding the errors $\delta_{V,l}^{(g),N}$ defined by

$$\delta_{V,l}^{(g),N} = \tilde{\mathcal{W}}_{V,l}^N - S_{V,l}^{(g),N}. \tag{3.33}$$

In particular, we shall set $\delta_{V,l}^{(-1),N} = \tilde{\mathcal{W}}_{V,l}^N$.

We will derive equations on the errors $\delta_{V,l}^{(g),N}$. To make this clearer, we first consider the case $g=0, l=1$. In that case, when $\|\boldsymbol{z}\|_\infty < R_V$, we have as a consequence of (3.22),

$$\frac{1}{2}\left(S_{V,1}^{(0),N} \otimes \mathcal{M}_{V,1}^{(0),N} + \mathcal{M}_{V,1}^{(0),N} \otimes S_{V,1}^{(0),N}\right)(\partial_i P) + S_1^{(0),N}((\mathcal{D}_i V)P) = 0\,. \tag{3.34}$$

On the other hand, Proposition 3.7.6 implies

$$\tilde{\mathcal{W}}_{V,1}^{N} \otimes \tilde{\mathcal{W}}_{V,1}^{N}(\partial_i P) + \tilde{\mathcal{W}}_{V,1}^{N}(\mathcal{D}_i V \cdot P) = -\frac{1}{N^2}\tilde{\mathcal{W}}_{V,2}^{N}(\partial_i P)\,. \tag{3.35}$$

Taking the difference of (3.34) and (3.35), we get

$$\begin{aligned}\frac{1}{2}(\delta_{V,1}^{(0),N} \otimes \tilde{\mathcal{W}}_{V,1}^{N} + \tilde{\mathcal{W}}_{V,1}^{N} \otimes \delta_{V,1}^{(0),N} + S_{V,1}^{(0),N} \otimes \delta_{V,1}^{(0),N} + \delta_{V,1}^{(0),N} \otimes S_{V,1}^{(0),N})(\partial_i P)\\ + \delta_{V,1}^{(0),N}((\mathcal{D}_i V)P) = -\frac{1}{N^2}\tilde{\mathcal{W}}_{V,2}^{N}(\partial_i P)\,.\end{aligned}$$

We apply the gradient trick as in Section 3.9. To do so, we replace P by $\mathcal{D}_i P$. Making use of Lemma 3.9.1 and the fact that

$$\frac{1}{2}\left(\delta_{V,1}^{(0),N} \otimes \tilde{\mathcal{W}}_{V,1}^{N} + \tilde{\mathcal{W}}_{V,1}^{N} \otimes \delta_{V,1}^{(0),N}\right) \quad \text{and} \quad \frac{1}{2}\left(S_{V,1}^{(0),N} \otimes \delta_{V,1}^{(0),N} + \delta_{V,1}^{(0),N} \otimes S_{V,1}^{(0),N}\right)$$

are symmetric, we can make the master operator defined in Definition 3.7.8 appear. We get

$$\delta_{V,1}^{(0),N}\left(\Xi^{V}_{\tilde{\mathcal{W}}_{V,1}^{N}/2 + S_{V,1}^{(0),N}/2}\right)(P) = -\delta_{V,1}^{(0),N} \otimes \delta_{V,1}^{(0),N}(\bar{\Delta} P) - \frac{1}{N^2}\tilde{\mathcal{W}}_{V,2}^{N}(\partial_i P)\,. \tag{3.36}$$

We now turn to the general case. When $\|\boldsymbol{z}\|_\infty < R_V$, the truncated formal cumulants satisfy the equation

$$\begin{aligned}&\sum_{\substack{I\subset[l-1]\\ 0\le f\le g}} \frac{1}{2}\left(\frac{1}{N^{2f}}\mathcal{M}_{V,|I|+1}^{(f),N} \otimes S_{V,|I^c|+1}^{(g-f),N} + \frac{1}{N^{2f}}S_{V,|I|+1}^{(g-f),N} \otimes \mathcal{M}_{V,|I^c|+1}^{(f),N}\right)(\boldsymbol{P}_I \otimes \boldsymbol{P}_{I^c} \# \partial_i P_l)\\ &+ S_l^{(g),N}(\boldsymbol{P}_{[l-1]} \otimes (\mathcal{D}_i V)P_l)\\ &= -\frac{1}{N^2}S_{V,l}^{(g-1),N}(\boldsymbol{P}_{[l-1]} \otimes \partial_i P_l) - \sum_{j=1}^{l-1} S_{V,l-1}^{(g),N}(P_1 \otimes \cdots P_{j-1} \otimes P_j \otimes P_{l-1} \otimes (\mathcal{D}_i P_j)P_l)\,.\end{aligned} \tag{3.37}$$

These equations are obtained by summing the equations (3.29) for different values of g, multiplied by $1/N^{2g}$.

Together with (3.31), these equations imply equation (3.38) on the errors defined by (3.33). Before stating the equation, we explain how this equation is derived. We subtract from (3.31) equation (3.37). To have all the terms from (3.31) simplify, we must rewrite the most complicated term

$$\sum_{\substack{I\subset[l-1]\\ 0\le f\le g}} \frac{1}{2}\left(\frac{1}{N^{2f}}\mathcal{M}_{V,|I|+1}^{(f),N} \otimes S_{V,|I^c|+1}^{(g-f),N} + \frac{1}{N^{2f}}S_{V,|I|+1}^{(g-f),N} \otimes \mathcal{M}_{V,|I^c|+1}^{(f),N}\right)(\boldsymbol{P}_I \otimes \boldsymbol{P}_{I^c} \# \partial_i P_l)\,.$$

We rewrite $S_{V,l}^{(g),N} = \tilde{\mathcal{W}}_{V,l}^{N} - \delta_{V,l}^{(g),N}$ in the sum

$$\begin{aligned}\sum_{0\le f\le g} \frac{1}{N^{2f}}\mathcal{M}_{V,|I|+1}^{(f),N} \otimes S_{V,|I^c|+1}^{(g-f),N} &= \sum_{0\le f\le g} \frac{1}{N^{2f}}\mathcal{M}_{V,|I|+1}^{(f),N} \otimes \left(\tilde{\mathcal{W}}_{V,|I^c|+1}^{N} - \delta_{V,|I^c|+1}^{(g-f),N}\right)\\ &= S_{V,|I|+1}^{(g),N} \otimes \tilde{\mathcal{W}}_{V,|I^c|+1}^{N} - \sum_{0\le f\le g} \frac{1}{N^{2f}}\mathcal{M}_{V,|I|+1}^{(f),N} \otimes \delta_{V,|I^c|+1}^{(g-f),N}\\ &= \tilde{\mathcal{W}}_{V,|I|+1}^{N} \otimes \tilde{\mathcal{W}}_{V,|I^c|+1}^{N} - \delta_{V,|I|+1}^{(g),N} \otimes \tilde{\mathcal{W}}_{V,|I^c|+1}^{N} - \sum_{0\le f\le g} \frac{1}{N^{2f}}\mathcal{M}_{V,|I|+1}^{(f),N} \otimes \delta_{V,|I^c|+1}^{(g-f),N}\,.\end{aligned}$$

We proceed similarly for $\sum_{0\leq f\leq g}\frac{1}{N^{2f}}S^{(g-f),N}_{V,|I^c|+1}\otimes\mathcal{M}^{(f),N}_{V,|I|+1}$. After this rewriting and subtracting (3.31), we have

$$\begin{aligned}
&\sum_{I\subset[l-1]}\frac{1}{2}\left(\delta^{(g),N}_{V,|I|+1}\otimes\tilde{\mathcal{W}}^N_{V,|I^c|+1}+\tilde{\mathcal{W}}^N_{V,|I|+1}\otimes\delta^{(g),N}_{V,|I^c|+1}\right)(\boldsymbol{P}_I\otimes\boldsymbol{P}_{I^c}\#\partial_i P_l)\\
&+\sum_{\substack{I\subset[l-1]\\0\leq f\leq g}}\frac{1}{2}\left(\frac{1}{N^{2f}}\mathcal{M}^{(f),N}_{V,|I|+1}\otimes\delta^{(g-f),N}_{V,|I^c|+1}+\frac{1}{N^{2f}}\delta^{(g-f),N}_{V,|I|+1}\otimes\mathcal{M}^{(f),N}_{V,|I^c|+1}\right)(\boldsymbol{P}_I\otimes\boldsymbol{P}_{I^c}\#\partial_i P_l)\\
&\quad+\delta^{(g),N}_{V,l}(\boldsymbol{P}_{[l-1]}\otimes(\mathcal{D}_iV)P_l)\\
&\quad=-\frac{1}{N^2}\delta^{(g-1),N}_{V,l+1}(\boldsymbol{P}_{[l-1]}\otimes\partial_i P_l)-\sum_{j=1}^{l-1}\delta^{(g),N}_{V,l-1}(P_1\otimes\cdots P_{j-1}\otimes P_j\otimes P_{l-1}\otimes(\mathcal{D}_iP_j)P_l)\,.
\end{aligned}\tag{3.38}$$

Using the gradient trick (see Section 3.9), these equations can be rewritten as follows

$$\begin{aligned}
&\delta^{(g),N}_{V,l}\left(\boldsymbol{P}_{[l-1]}\otimes\Xi^V_{\tilde{\mathcal{W}}^N_{V,1}/2+\mathcal{M}^{(0),N}_{V,1}/2}\right)(P_l)\\
&=-\frac{1}{N^2}\delta^{(g-1)}_{l+1}(\boldsymbol{P}_{[l-1]}\otimes\bar{\Delta}P_l)\\
&-\frac{1}{2}\big([\tilde{\mathcal{W}}^N_{V,l}+\mathcal{M}^{(0),N}_{V,l}](\boldsymbol{P}_{[l-1]}\otimes\mathrm{Id})\otimes\delta^{(g),N}_{V,1}\\
&\qquad\qquad+\delta^{(g),N}_{V,1}\otimes[\tilde{\mathcal{W}}^N_{V,l}+\mathcal{M}^{(0),N}_{V,l}](\boldsymbol{P}_{[l-1]}\otimes\mathrm{Id})\big)(\bar{\Delta}P_l)\\
&-\sum_{\emptyset\subsetneq I\subsetneq[l-1]}\frac{1}{2}\Big([\tilde{\mathcal{W}}^N_{V,l}+\mathcal{M}^{(0),N}_{V,l}]\otimes\delta^{(g),N}_{V,|I^c|+1}\\
&\qquad\qquad+\delta^{(g),N}_{V,|I|+1}\otimes[\tilde{\mathcal{W}}^N_{V,l}+\mathcal{M}^{(0),N}_{V,l}]\Big)(\boldsymbol{P}_I\otimes\boldsymbol{P}_{I^c}\#\bar{\Delta}P_l)\\
&-\sum_{\substack{I\subset[l-1]\\1\leq f\leq g}}\frac{1}{2}\left(\frac{1}{N^{2f}}\mathcal{M}^{(f),N}_{V,|I|+1}\otimes\delta^{(g-f),N}_{V,|I^c|+1}+\frac{1}{N^{2f}}\delta^{(g-f),N}_{V,|I|+1}\otimes\mathcal{M}^{(f),N}_{V,|I^c|+1}\right)(\boldsymbol{P}_I\otimes\boldsymbol{P}_{I^c}\#P_l)\\
&-\sum_{j=1}^{l-1}\delta^{(g)}_{l-1}(P_1\otimes\cdots\otimes\check{P}_j\otimes\cdots\otimes P_{l-1}\otimes\bar{\mathcal{P}}^{P_j}P_l).
\end{aligned}\tag{3.39}$$

The notation Id in the third line means the identity operator, in particular terms on the third line must be read as follows:

$$[\tilde{\mathcal{W}}^N_{V,l}+\mathcal{M}^{(0),N}_{V,l}](\boldsymbol{P}_{[l-1]}\otimes\mathrm{Id})\otimes\delta^{(g),N}_{V,1}(P\otimes Q)=[\tilde{\mathcal{W}}^N_{V,l}+\mathcal{M}^{(0),N}_{V,l}](\boldsymbol{P}_{[l-1]}\otimes P)\otimes\delta^{(g),N}_{V,1}(Q)\,.$$

The bounds of Proposition 3.5.22 imply the following results.

Lemma 3.7.10. *Assume that for all $N\geq 1$, $\operatorname{Tr}V$ is real and $\|A^N_i\|\leq 1$ for all i (Hypotheses 3.1.1 and 3.1.3). There exists $\xi>1$ and $\epsilon>0$, such that if*

$$\|\boldsymbol{z}\|_\infty<\epsilon,$$

then, that for all $g\geq 0$ and $l\geq 1$, we have

$$\|\delta^{(0)}_1\|_\xi\leq\frac{C}{N^2}.$$

Proof. Consider the equation (3.36) for the errors with $g=0,l=1$:

$$\delta^{(0)}_1\left(\Xi^V_{\tilde{\mathcal{W}}^N_{V,1}/2+\mathcal{M}^{(0),N}_{V,1}/2}P\right)=-\delta^{(0)}_1\otimes\delta^{(0)}_1(\bar{\Delta}P)-\frac{1}{N^2}\tilde{\mathcal{W}}^N_{V,2}(\bar{\Delta}P)\,.$$

First, notice that the series $\mathcal{M}_{V,1}^{(0),N}$ satisfies $\|\mathcal{M}_{V,1}^{(0),N}\|_{4A_k} \leq \frac{2^k}{B_k}$, with A_k and B_k the constants from Proposition 3.5.22. Proposition 3.9.3 implies that

$$\|\bar{T}_{\mathcal{M}_{V,1}^{(0),N}}\|_\xi \leq \frac{2^{k+1}}{B_k}\frac{4A_k}{\xi+4A_k}.$$

Let $\xi \geq 32A_k \geq 12$ and $0 < \epsilon < R_V$, such that

$$K(\xi, V) = 2\frac{\xi+1}{\xi(\xi-1)} + \frac{2^k}{B_k}\frac{4A_k}{\xi+4A_k} + \|\Pi V\|_1 \nu \xi^\nu < 1/2.$$

In that case, the operator $\Xi^V_{\tilde{\mathcal{W}}_{V,1}^N/2+\mathcal{M}_{V,1}^{(0),N}/2} : \mathcal{B}_\xi^\perp \to \mathcal{B}_\xi^\perp$ is invertible.

We get that

$$\|\delta_1^{(0)}\|_\xi \leq \|\delta_1^{(0)}\|_\xi \|\bar{T}_{\delta_1^{(0)}}\|_\xi \|(\Xi^V_{\tilde{\mathcal{W}}_{V,1}^N})^{-1}\|_\xi + \frac{C'}{N^2}\|\bar{\Delta}\|_{\xi,\xi/2}\|(\Xi^V_{\tilde{\mathcal{W}}_{V,1}^N})^{-1}\|_\xi,$$

where we used that there exists a constant $C' > 0$ such that $\|\tilde{\mathcal{W}}_{V,2}^N\|_{\xi/2} \leq C'$ by [GN15, Theorem 22]. Note that for this Theorem 22 to be applicable, one must show that the sequence of cumulants is ξ-uniformly bounded in the sense of [GN15, Definition 21]. This is shown assuming Assumption 3.1.1 in [GN15, Corollary 32] for all $\xi \geq 12$.

The bound on $\mathcal{M}_{V,1}^{(0),N}$ implies that $\|\delta_1^{(0)}\|_{4A_k} \leq 1 + \frac{2^k}{B_k} \leq 2$. This fact and Proposition 3.9.3 give

$$\|\bar{T}_{\delta_1^{(0)}}\|_\xi \leq 2(1+\frac{2^k}{B_k})\frac{4A_k}{\xi-4A_k} < 1/2.$$

With this result and [GN15, Proposition 19], we finally get that

$$\|\delta_1^{(0)}\|_\xi \leq \frac{C'}{N^2}\frac{\|(\Xi^V_{\tilde{\mathcal{W}}_{V,1}^N})^{-1}\|_\xi}{1-\|\bar{T}_{\delta_1^{(0)}}\|_\xi\|(\Xi^V_{\tilde{\mathcal{W}}_{V,1}^N})^{-1}\|_\xi} \leq \frac{C'}{N^2}\frac{1}{1/2-K(\xi,V)}.$$

□

Proposition 3.7.11. *Assume that for all $N \geq 1$, $\mathrm{Tr}\, V$ is real and $\|A_i^N\| \leq 1$ for all i (Hypotheses 3.1.1 and 3.1.3). There exists $\xi > 1$ and $\epsilon > 0$, such that if*

$$\|z\|_\infty < \epsilon,$$

then for all $g \geq 0$ and $l \geq 1$, we have

$$\|\delta_l^{(g)}\|_{2^{l-2}\xi} = \mathcal{O}\big(N^{-2g-2}\big).$$

Proof. We proceed by induction on (g, l), with lexicographic order. For $l = 1, g = 0$, the result is given by Lemma 3.7.10. Assume now that for all $(g', l') < (g, l)$, we have

$$\|\delta_{l'}^{(g')}\|_\xi = \mathcal{O}\Big(N^{-2g'-2}\Big).$$

Then, in the equations (3.39) for the errors, all the terms on the right side of the equation are of order N^{-2g}. Note that terms $\delta_l^{(-1)} = \tilde{\mathcal{W}}_{V,l}^N$ are bounded using [GN15, Theorem 22]. This gives the result. □

We can finally prove Theorem 3.1.4.

Proof of Theorem 3.1.4. Proposition 3.7.11 directly implies Theorem 3.1.4, with Hypothesis 3.1.2 replaced by Hypothesis 3.1.3. Then, if we only assume Hypothesis 3.1.2, we set

$$c = \frac{1}{\sup_{N\geq 1}\sup_{1\leq i\leq p}\|A_i^N\|}.$$

We can then replace each matrix A_i^N by cA_i^N and rescale each coefficient z_i of V by an appropriate multiple of c^{-1}. When the new coefficients of V are small enough, we can apply the result obtained with Hypothesis 3.1.2 and obtain the result. □

Remark 3.7.12. Notice that for N big enough, the series

$$\sum_{h=0}^{g}\frac{1}{N^{2h}}\mathcal{M}_{V,l}^{(h),N}(P_1,\dots,P_l)$$

is well defined for all V with $\boldsymbol{z}$ small enough, even if Hypothesis 3.1.1 is not satisfied. In fact, for any V with $\boldsymbol{z}$ small, provided the cumulants exist, are bounded, and satisfy the Dyson-Schwinger equations, the same method applies and the asymptotic topological expansion holds.

The complex asymptotics of the HCIZ and BGW partition functions were studied with a different method in [Nov20].

3.8 Bounds for the sum of maps $\mathcal{M}_{0,l}^{(g),N}$ (Proof of Proposition 3.5.22)

This Section gives a detailed proof of Proposition 3.5.22.

We assume that $\nu\geq 1$ and that up to cyclic permutation of its factors we can write P_l as Pu. If P_l has no term u, a similar argument holds with $P_l = u^*P$. Furthermore, to make notation less cumbersome, we write

$$\mathcal{M}_{\boldsymbol{n},l}^{(g),N}(P_1,\dots,P_l) = \mathcal{M}_{0,\sum_i n_i+l}^{(g),N}(\underbrace{q_1,\dots,q_1}_{n_1\text{ times}},\dots,\underbrace{q_k,\dots,q_k}_{n_k\text{ times}},P_1,\dots,P_l),$$

and omit the indices k and ν in the constants.

To prove the result, we do an induction on $\mathbf{N}^{k+3}$, where we endow a tuple $(g,l,n_1,\dots,n_k,m)$ with the lexicographic order. Notice that the result is obvious when $n_1=\dots=n_k=0$, $g=0$, $l=1$ when $\deg\boldsymbol{P}=1$ (as $\mathcal{M}_{0,1}^{(0),N}(MU^{\pm1})=0$ for all $M\in\mathcal{Y}$), and when $m=1=\frac{1}{2}\deg\boldsymbol{P}$ (as $\mathcal{M}_{0,1}^{(0),N}(M_1UM_2U^{-1})=\operatorname{tr}M_1\operatorname{tr}M_2$), as soon as $M\geq 1$. In fact, by Lemma 3.3.30, $|\mathcal{M}_{\boldsymbol{0},1}^{(0),N}(P)|\leq 1$ for all $P\in\mathcal{X}$.

Assuming that $m\geq 2$ or $\boldsymbol{n}\neq 0$, Theorem 3.5.1 yields

$$\begin{aligned}\frac{1}{\boldsymbol{n}!}\mathcal{M}_{\boldsymbol{n},l}^{(g),N}(P_1,\dots,P_{l-1},Pu) &= -\frac{1}{\boldsymbol{n}!}\mathcal{M}_{\boldsymbol{n},l+1}^{(g-1),N}(P_1,\dots,P_{l-1},(\partial P)\times 1\otimes u)\\ &\quad-\sum_{\substack{I\subset[l-1]\\ \boldsymbol{n_1}+\boldsymbol{n_2}=\boldsymbol{n}}}\sum_{g_1+g_2=g}\frac{1}{\boldsymbol{n_1}!}\frac{1}{\boldsymbol{n_2}!}\mathcal{M}_{\boldsymbol{n_1},|I|+1}^{(g_1),N}\otimes\mathcal{M}_{\boldsymbol{n_2},|I^c|+1}^{(g_2),N}(\boldsymbol{P}_I\otimes\boldsymbol{P}_{I^c}\#(\partial P)\times 1\otimes u)\\ &\quad-\sum_{j=1}^{l-1}\frac{1}{\boldsymbol{n}!}\mathcal{M}_{\boldsymbol{n},l-1}^{(g),N}(P_1,\dots,\check{P}_j,\dots,P_{l-1},(\mathcal{D}P_j)Pu)\\ &\quad-\sum_{j=1}^{k}\frac{1}{(\boldsymbol{n}-1_j)!}\mathcal{M}_{\boldsymbol{n}-1_j,l}^{(g),N}(P_1,\dots,P_{l-1},(\mathcal{D}q_j)Pu),\end{aligned}\tag{3.40}$$

where $\check{P}_j$ means that P_j is removed.

Now assuming that the bound (3.21) holds for $(g',l',n'_1,\dots,n'_k,m')<(g,l,n_1,\dots,n_k,m)$, we get four terms from (3.40).

1.

$$
\begin{aligned}
&\frac{1}{\boldsymbol{n}!}|\mathcal{M}_{\boldsymbol{n},l+1}^{(g-1),N}(P_1,\dots,P_{l-1},(\partial P)\times 1\otimes u)| \\
&\quad\le \sum_{m'=1}^{\deg Pu} A^{(l+1)(2m+\nu\boldsymbol{n})}B^{-l-1}C^{(g-1)(2m+\nu\boldsymbol{n})}D^{\boldsymbol{n}} \\
&\qquad\qquad \times \operatorname{Cat}_{m'}\operatorname{Cat}_{\deg Pu-m'}\prod_{i=1}^{l-1}\operatorname{Cat}_{\deg P_i}\prod_{j=1}^{k}\operatorname{Cat}_{n_j} \\
&\quad\le A^{(l+1)(2m+\nu\boldsymbol{n})}B^{-l-1}C^{(g-1)(2m+\nu\boldsymbol{n})}D^{\boldsymbol{n}} \\
&\qquad\qquad \times (\operatorname{Cat}_{\deg Pu+1}-\operatorname{Cat}_{\deg Pu})\prod_{i=1}^{l-1}\operatorname{Cat}_{\deg P_i}\prod_{j=1}^{k}\operatorname{Cat}_{n_j} \\
&\quad\le \frac{3}{B}\left(\frac{A}{C}\right)^{2m}A^{l(2m+\nu\boldsymbol{n})}B^{-l}C^{g(2m+\nu\boldsymbol{n})}D^{\boldsymbol{n}}\prod_{i=1}^{l}\operatorname{Cat}_{\deg P_i}\prod_{j=1}^{k}\operatorname{Cat}_{n_j}\,.
\end{aligned}
$$

In the second line, we expanded the non-commutative derivative (see Definition 3.5.2). In the third line we used the recurrence formula for Catalan numbers $\operatorname{Cat}_{n+1}=\sum_{i=0}^{n}\operatorname{Cat}_i\operatorname{Cat}_{n-i}$ and in the fourth line we used that $\operatorname{Cat}_{n+1}\le 4\operatorname{Cat}_n$ for all $n\in\mathbf{N}$. We choose A and C so that $A/C\le 1$ and $B\ge 12$.

2.

$$
\begin{aligned}
&\sum_{\substack{I\subset[l-1]\\ \boldsymbol{n_1}+\boldsymbol{n_2}=\boldsymbol{n}}}\sum_{g_1+g_2=g}\frac{1}{\boldsymbol{n_1}!}\frac{1}{\boldsymbol{n_2}!}|\mathcal{M}_{\boldsymbol{n_1},|I|+1}^{(g_1),N}\otimes\mathcal{M}_{\boldsymbol{n_2},|I^c|+1}^{(g_2),N}|(\boldsymbol{P}_I\otimes\boldsymbol{P}_{I^c}\#(\partial P)\times 1\otimes u) \\
&\le \sum_{m'=1}^{\deg P}\sum_{I\subset[l-1]}\sum_{g_1+g_2=g}A^{m_1(|I|+1)+m_2(|I^c|+1)}B^{-l-1}C^{g_1m_1+g_2m_2}\operatorname{Cat}_{m'}\operatorname{Cat}_{\deg Pu-m'} \\
&\qquad \times \sum_{\boldsymbol{n_1}+\boldsymbol{n_2}=\boldsymbol{n}}A^{l\nu\boldsymbol{n}}C^{g\nu\boldsymbol{n}}D^{\boldsymbol{n}}\prod_{i=1}^{l-1}\operatorname{Cat}_{\deg P_i}\prod_{i=1}^{k}\operatorname{Cat}_{n_{1,i}}\operatorname{Cat}_{n_{2,i}},
\end{aligned}
$$

where we used the notation $m_1=\sum_{i\in I}\deg P_i+m'$ and $m_2=\sum_{i\in I^c}\deg P_i+\deg Pu-m'$. With this notation, we get as soon as $C\ge 2$,

$$
\begin{aligned}
\sum_{g_1+g_2=g}C^{g_1m_1+g_2m_2} &= C^{2mg}\sum_{h=0}^{g}\left(\frac{1}{C^{m_2}}\right)^{h}\left(\frac{1}{C^{m_1}}\right)^{g-h} \\
&\le C^{2mg}\sum_{h=0}^{g}2^{-g} \\
&\le C^{2mg}\,.
\end{aligned}
$$

In the second line, we used that $m_1,m_2\ge 1$. Similarly, we have when $A\ge 2$,

$$
\begin{aligned}
\sum_{I\subset[l-1]}A^{m_1(|I|+1)+m_2(|I^c|+1)} &= A^{2ml}\sum_{I\subset[l-1]}A^{-m_1|I^c|-m_2|I|} \\
&\le A^{2ml}\sum_{i=0}^{l-1}\binom{l-1}{i}\left(\frac{1}{A^{\deg Pu-m'}}\right)^{i}\left(\frac{1}{A^{m'}}\right)^{l-i-i} \\
&= A^{2ml}\left(\frac{1}{A^{\deg Pu-m'}}+\frac{1}{A^{m'}}\right)^{l-1} \\
&\le A^{2ml}.
\end{aligned}
$$

We finally get

$$\sum_{\substack{I\subset[l-1]\\ \boldsymbol{n_1}+\boldsymbol{n_2}=\boldsymbol{n}}}\sum_{g_1+g_2=g}\frac{1}{\boldsymbol{n_1}!}\frac{1}{\boldsymbol{n_2}!}|\mathcal{M}^{(g_1),N}_{\boldsymbol{n_1},|I|+1}\otimes\mathcal{M}^{(g_2),N}_{\boldsymbol{n_2},|I^c|+1}|(\boldsymbol{P}_I\otimes\boldsymbol{P}_{I^c}\#(\partial P)\times 1\otimes u)$$
$$\leq\frac{6\cdot 4^k}{B}A^{l(2m+\nu\boldsymbol{n})}B^{-l}C^{g(2m+\nu\boldsymbol{n})}D^{\boldsymbol{n}}\prod_{i=1}^{l}\mathrm{Cat}_{\deg P_i}\prod_{i=1}^{k}\mathrm{Cat}_{n_i}$$

Thus, we choose $B\geq 6\cdot 4^{k+1}$.

3.

$$\sum_{j=1}^{l-1}\frac{1}{\boldsymbol{n}!}|\mathcal{M}^{(g),N}_{\boldsymbol{n},l-1}(P_1,\ldots,\check{P}_j,\ldots,P_{l-1},(\mathcal{D}P_j)Pu)|$$
$$\leq\sum_{j=1}^{l-1}(\deg P_j)A^{(l-1)(2m+\nu\boldsymbol{n})}B^{-l+1}C^{g(2m+\nu\boldsymbol{n})}D^{\boldsymbol{n}}$$
$$\times\,\mathrm{Cat}_{\deg P_j+\deg Pu}\prod_{\substack{i=1\\ i\neq j}}^{l-1}\mathrm{Cat}_{\deg P_i}\prod_{j=1}^{k}\mathrm{Cat}_{n_j}$$
$$\leq\frac{B}{A^{2m+\nu\boldsymbol{n}}}\left(\sum_{j=1}^{l-1}(\deg P_j)\frac{\mathrm{Cat}_{\deg P_j+\deg Pu}}{\mathrm{Cat}_{\deg P_j}\mathrm{Cat}_{\deg Pu}}\right)A^{l(2m+\nu\boldsymbol{n})}B^{-l}C^{g(2m+\nu\boldsymbol{n})}D^{\boldsymbol{n}}$$
$$\times\prod_{i=1}^{l}\mathrm{Cat}_{\deg P_i}\prod_{j=1}^{k}\mathrm{Cat}_{n_j}\,.$$

To bound this term, we use the following estimate for the Catalan numbers, a consequence of the Stirling bound

$$\frac{4^n}{(n+1)\sqrt{\pi n}}\exp\!\left(\frac{1}{24n+1}-\frac{1}{24n}\right)\leq\mathrm{Cat}_n\leq\frac{4^n}{(n+1)\sqrt{\pi n}}\exp\!\left(\frac{1}{24n}-\frac{1}{24n+2}\right),$$

which implies

$$\frac{4^n}{\sqrt{\pi}(n+1)^{3/2}}\leq\mathrm{Cat}_n\leq\frac{4^n}{\sqrt{\pi}n^{3/2}}.$$

It implies that for $p,q\in\mathbf{N}^*$,

$$\frac{\mathrm{Cat}_{p+q}}{\mathrm{Cat}_p\,\mathrm{Cat}_q}\leq\pi^{1/2}\left(\frac{(p+1)(q+1)}{p+q}\right)^{3/2}\leq\pi^{1/2}(p+1)^{3/2}.$$

Thus,

$$\frac{B}{A^{2m+\nu\boldsymbol{n}}}\left(\sum_{j=1}^{l-1}(\deg P_j)\frac{\mathrm{Cat}_{\deg P_j+\deg Pu}}{\mathrm{Cat}_{\deg P_j}\mathrm{Cat}_{\deg Pu}}\right)\leq\frac{\pi^{1/2}B}{A^{2m+\nu\boldsymbol{n}}}(\deg Pu+1)^{3/2}(2m-\deg Pu).$$

As we can assume that $m\geq 1$ (else this term could be bounded by 0), it suffices to choose $A\leq 2B^{1/2}\pi^{1/4}2^{3/2}$. Notice that for all $n\geq 1$, $(n+1)^{3/2}\leq 2^{3n/2}$.

4.

$$\sum_{j=1}^{k}\frac{1}{(\boldsymbol{n}-1_j)!}\mathcal{M}^{(g),N}_{\boldsymbol{n}-1_j,l}(P_1,\dots,P_{l-1},(\mathcal{D}q_j)Pu)$$
$$\leq \frac{1}{D}\sum_{j=1}^{k}(\deg q_k)A^{l(2m+\nu\boldsymbol{n})}B^{-l}C^{g(2m+\nu\boldsymbol{n})}D^{\boldsymbol{n}}$$
$$\times\frac{\mathrm{Cat}_{\deg Pu+\deg q_j}\mathrm{Cat}_{n_j-1}}{\mathrm{Cat}_{\deg Pu}\mathrm{Cat}_{n_j}}\prod_{i=1}^{l}\mathrm{Cat}_{\deg P_i}\prod_{i=1}^{k}\mathrm{Cat}_{n_i}$$
$$\leq \frac{1}{D}\sum_{j=1}^{k}4^{\deg q_k}(\deg q_k)A^{l(2m+\nu\boldsymbol{n})}B^{-l}C^{g(2m+\nu\boldsymbol{n})}D^{\boldsymbol{n}}\prod_{i=1}^{l}\mathrm{Cat}_{\deg P_i}\prod_{i=1}^{k}\mathrm{Cat}_{n_i}$$
$$\leq \frac{1}{D}\sum_{j=1}^{k}(4\mathrm{e}^{1/\mathrm{e}})^{\deg q_k}A^{l(2m+\nu\boldsymbol{n})}B^{-l}C^{g(2m+\nu\boldsymbol{n})}D^{\boldsymbol{n}}\prod_{i=1}^{l}\mathrm{Cat}_{\deg P_i}\prod_{i=1}^{k}\mathrm{Cat}_{n_i}\,.$$

We choose $D = 4k(4\mathrm{e}^{1/\mathrm{e}})^{\nu}$ to get the result. Notice that we can thus choose

$$A = C = 2^{k+3}\sqrt{6}\pi^{1/4}$$
$$B = 3\cdot 4^{k+1}$$
$$D = 4k(4\mathrm{e}^{1/\mathrm{e}})^{\nu}.$$

3.9 The gradient trick

We use several times the gradient trick, previously introduced in [GN15]. We recall it here for the convenience of the reader. The main idea of the gradient trick is to replace the polynomial P (or P_l) the equations of Proposition 3.7.6 (or in the Dyson-Schwinger problem (3.30), see Section 3.7) by its cyclic derivative $\mathcal{D}_iP$. An operator – the master operator introduced below – naturally appears in the equations. When the potential V is small enough, this operator is invertible. The gradient trick was introduced in [GN15] to study the Dyson-Schwinger lattice of equations.

3.9.1 The trick

The gradient trick allows us to simplify quadratic terms. We take as an example the equation for the sums of maps for $g = 0, l = 2$

$$\sum_{I\subset[l-1]}\mathcal{M}^{(0),N}_{0,|I|+1}\otimes\mathcal{M}^{(0),N}_{0,|I^c|+1}(\boldsymbol{P}_I\otimes\boldsymbol{P}_{I^c}\#\partial_iP_2) = -\mathcal{M}^{(g),N}_{0,1}((\mathcal{D}_iP_1)P_2).$$

We can rewrite it as

$$\mathcal{M}^{(0),N}_{0,2}(P_1\otimes\mathrm{Id}\otimes\mathcal{M}^{(0),N}_{0,1}+P_1\otimes\mathcal{M}^{(0),N}_{0,1}\otimes\mathrm{Id})(\partial_iP_2) = -\mathcal{M}^{(g),N}_{0,1}((\mathcal{D}_iP_1)P_2)\,,$$

where Id denote the identity operator. In particular, the notation $P_1\otimes\mathrm{Id}\otimes\mathcal{M}^{(0),N}_{0,1}(\partial_iP_2)$ must be understood as follows. For $(Q_1,Q_2)\in\mathcal{A}_n^2$,

$$P_1\otimes\mathrm{Id}\otimes\mathcal{M}^{(0),N}_{0,1}(Q_1\otimes Q_2) = (\mathcal{M}^{(0),N}_{0,1}(Q_2))\cdot(P_1\otimes Q_1)\in\mathcal{A}_n^{\otimes 2}\,.$$

We now replace P_2 by its cyclic derivative $\mathcal{D}_iP_2$, and obtain

$$\mathcal{M}^{(0),N}_{0,2}(P_1\otimes\mathrm{Id}\otimes\mathcal{M}^{(0),N}_{0,1}+P_1\otimes\mathcal{M}^{(0),N}_{0,1}\otimes\mathrm{Id})(\partial_i\mathcal{D}_iP_2) = -\mathcal{M}^{(g),N}_{0,1}((\mathcal{D}_iP_1)(\mathcal{D}_iP_2))\,.$$

Lemma 3.9.1. *Let $\mu_2\colon \mathcal{A}_n \times \mathcal{A}_n \to \mathbb{C}$ be a bilinear form, tracial in each of its variables. For a monomial $P \in \mathcal{X}_n$, write $\deg_i^+(P)$ for the number of factors u_i in P and $\deg_i^-(P)$ for the number of factors u_i^* in P. We have for any monomial $P \in \mathcal{A}_n$,*

$$\mu_2(\partial_i \mathcal{D}_i P) = \deg_i^+(P)\mu_2(P \otimes 1) + \deg_i^-(P)\mu_2(1 \otimes P) + \mu_2(\Delta_i P)\,,$$

with the operator Δ_i introduced in Definition 3.7.7. In particular, if μ_2 is symmetric, we get

$$\mu_2(\partial_i \mathcal{D}_i P) = \deg_i(P) \cdot \mu_2(1 \otimes P) + \mu_2(\Delta_i P)\,.$$

This Lemma allows us to rewrite the above expression as

$$\mathcal{M}_{0,2}^{(0),N}\big(\deg_i(P_2)P_1 \otimes \mathrm{Id} + (P_1 \otimes \mathrm{Id} \otimes \mathcal{M}_{0,1}^{(0),N} + P_1 \otimes \mathcal{M}_{0,1}^{(0),N} \otimes \mathrm{Id})(\Delta_i P_2)\big) = -\mathcal{M}_{0,1}^{(g),N}((\mathcal{D}_i P_1)(\mathcal{D}_i P_2)).$$

Introducing the operator

$$\mathcal{P}_i^q P = (\mathcal{D}_i q)(\mathcal{D}_i P),$$

for $P, q \in \mathcal{A}_n$, we get

$$\mathcal{M}_{0,2}^{(0),N}\big(\deg_i(P_2)P_1 \otimes \mathrm{Id} + (P_1 \otimes \mathrm{Id} \otimes \mathcal{M}_{0,1}^{(0),N} + P_1 \otimes \mathcal{M}_{0,1}^{(0),N} \otimes \mathrm{Id})(\Delta_i P_2)\big) = -\mathcal{M}_{0,1}^{(g),N}(\mathcal{P}_i^{P_1} P_2).$$

for $1 \leq i \leq n$.

Using the operators defined in Definition 3.7.7. The sum of maps $\mathcal{M}_{0,2}^{(0),N}$ satisfies

$$\mathcal{M}_{0,2}^{(0),N}\left(P_1 \otimes \mathrm{Id} + (P_1 \otimes \mathrm{Id} \otimes \mathcal{M}_{0,1}^{(0),N} + P_1 \otimes \mathcal{M}_{0,1}^{(0),N} \otimes \mathrm{Id})(\bar{\Delta} P_2)\right) = -\mathcal{M}_{0,1}^{(g),N}(\bar{\mathcal{P}}^{P_1} P_2)$$

for all $P_1, P_2 \in \mathcal{A}_n$. This computation justifies the introduction of the master operator Ξ_τ^V of Definition 3.7.8.

Thus, we have

$$\mathcal{M}_{0,2}^{(0),N}\left(P_1 \otimes \Xi^0_{\mathcal{M}_{0,1}^{(0),N}} P_2\right) = -\mathcal{M}_{0,1}^{(g),N}(\bar{\mathcal{P}}^{P_1} P_2)$$

for all $P_1, P_2 \in \mathcal{A}_n$. This will be called the secondary form of the equation (3.29). Notice that in this particular case $V = 0$. In the sequel, we will derive secondary equation with a potential.

3.9.2 Operator norm estimates

We now give some bounds on the norms of the different operators. These bounds and more were derived in [GN15, Section 3.2]. In particular, it was shown that under some hypotheses the master operator is invertible.

Proposition 3.9.2 ([GN15, Section 3.3])**.** *Let $\xi \geq 1$, $V \in \mathcal{A}$ and τ a tracial state satisfying $\|\tau\| \leq 1$. Introduce*

$$K(\xi, V) = 4\frac{\xi + 1}{\xi(\xi - 1)} + \|V\|_1 \xi^{\deg V} \deg V,$$

and assume that $K(\xi, V) < 1$. Then, the operator Ξ_τ^V extends to an operator $\mathcal{B}_\xi^\perp \to \mathcal{B}_\xi^\perp$ ($\mathcal{B}_\xi^\perp$ is defined in Definition 3.7.1) which is invertible, with inverse satisfying

$$\|\left(\Xi_\tau^V\right)^{-1}\|_\xi \leq \frac{1}{1 - K(\xi, V)}.$$

We use a slightly modified version of [GN15, Proposition 17].

Proposition 3.9.3. *Let $1 \leq \xi_1 < \xi_2$, and τ a linear form $\mathcal{A}_n \to \mathbb{C}$. We have*

$$\|\bar{T}_\tau\|_{\xi_2} \leq 2\|\tau\|_{\xi_1} \frac{\xi_1}{\xi_2 - \xi_1}.$$

Proof. We proceed as in [GN15]. Let P be a monomial of degree $d \geq 1$. We have

$$\begin{aligned} T_\tau P = &\sum_{i=1}^n \sum_{P = P_1 u_i P_2} \left(\sum_{P_2 P_1 u_i = Q_1 u_i Q_2 u_i} (Q_1 u_i \tau(Q_2 u_i) + \tau(Q_1 u_i) Q_2 u_i) \right) \\ &- \sum_{i=1}^n \sum_{P = P_1 u_i P_2} \left(\sum_{P_2 P_1 u_i = Q_1 u_i^{-1} Q_2 u_i} (Q_1 \tau(Q_2) + \tau(Q_1) Q_2) \right) \\ &- \sum_{i=1}^n \sum_{P = P_1 u_i^{-1} P_2} \left(\sum_{u_i^{-1} P_2 P_1 = u_i^{-1} Q_1 u_i Q_2} (Q_1 \tau(Q_2) + \tau(Q_1) Q_2) \right) \\ &+ \sum_{i=1}^n \sum_{P = P_1 u_i^{-1} P_2} \left(\sum_{u_i^{-1} P_2 P_1 = u_i^{-1} Q_1 u_i^{-1} Q_2} (u_i^{-1} Q_1 \tau(u_i^{-1} Q_2) + \tau(u_i^{-1} Q_1) u_i^{-1} Q_2) \right). \end{aligned}$$

We now take the norm $\| \cdot \|_{\xi_2}$ (recall Definition 3.7.1). Using the triangle inequality and $|\tau(P)| \leq \|\tau\|_{\xi_1} \xi_1^{\deg P}$, we get (as Q_1, Q_2 are monomials) that $\frac{\|T_\tau P\|_{\xi_2}}{\|\tau\|_{\xi_1}}$ is bounded by four terms of the form

$$\sum_{i=1}^n \sum_{P = P_1 u_i P_2} \left(\sum_{P_2 P_1 u_i = Q_1 u_i Q_2 u_i} (\xi_2^{\deg_i Q_1 u_i} \xi_1^{\deg_i Q_2 u_i} + \xi_1^{\deg_i Q_1 u_i} \xi_2^{\deg_i Q_2 u_i}) \right).$$

Grouping these terms together, we obtain

$$\frac{\|T_\tau P\|_{\xi_2}}{\|\tau\|_{\xi_1}} \leq 2 \sum_{i=1}^n \deg_i P \left(\sum_{k=1}^{\deg_i P - 1} \xi_2^k \xi_1^{\deg_i P - k} \right).$$

The factor $\deg_i P$ accounts for the first sum, on the decompositions of P as $P = P_1 u_i P_2$ or $P = P_1 u_i^* P_2$. The sum on k accounts from the decompositions of $P_2 P_1$ as $P_2 P_1 = Q_1 u_i Q_2$ or $P_2 P_1 = Q_1 u_i^* Q_2$. Finally, there is a factor 2 as we count those decompositions at most twice. For instance, we have

$$\begin{aligned} &\sum_{i=1}^n \sum_{P = P_1 u_i P_2} \left(\sum_{P_2 P_1 u_i = Q_1 u_i Q_2 u_i} (\xi_2^{\deg_i Q_1 u_i} \xi_1^{\deg_i Q_2 u_i} + \xi_1^{\deg_i Q_1 u_i} \xi_2^{\deg_i Q_2 u_i}) \right) \\ &+ \sum_{i=1}^n \sum_{P = P_1 u_i P_2} \left(\sum_{P_2 P_1 u_i = Q_1 u_i^{-1} Q_2 u_i} (\xi_2^{\deg_i Q_1} \xi_1^{\deg_i Q_2} + \xi_1^{\deg_i Q_1} \xi_2^{\deg_i Q_2}) \right) \\ &\leq \sum_{i=1}^n \sum_{P = P_1 u_i P_2} \sum_{P_2 P_1 = Q_1 u_i^{\pm 1} Q_2} 2 \xi_2^{\deg_i Q_1 u_i^{\pm 1}} \xi_1^{\deg_i Q_2 u_i^{\pm 1}} \\ &\leq \sum_{i=1}^n \sum_{P = P_1 u_i P_2} \sum_{k=1}^{\deg_i P - 1} 2 \xi_2^k \xi_1^{\deg_i P - 1} \\ &\leq 2 \sum_{i=1}^n \deg_i^+ P \sum_{k=1}^{\deg_i P - 1} \xi_2^k \xi_1^{\deg_i P - 1}. \end{aligned}$$

In the first inequality, we abused notation and wrote $u_i^{\pm 1}$ to mean either of u_i or u_i^*. In the last line, $\deg_i^+ P$ denotes the number of letter u_i in P.

We can then conclude:

$$\begin{aligned}
\frac{\|T_\tau P\|_{\xi_2}}{\|\tau\|_{\xi_1}} &= 2\sum_{i=1}^{n} (\deg_i P)\, \xi_2^{\deg_i P} \sum_{k=1}^{\deg_i P - 1} \left(\frac{\xi_1}{\xi_2}\right)^{\deg_i P - k} \\
&\leq 2\sum_{i=1}^{n} (\deg_i P)\, \xi_2^{\deg_i P} \frac{\xi_1}{\xi_2} \frac{1}{1 - \xi_1/\xi_2} \\
&\leq 2\sum_{i=1}^{n} (\deg_i P)\, \xi_2^{\deg_i P} \frac{\xi_1}{\xi_2 - \xi_1} \\
&\leq 2d \frac{\xi_1}{\xi_2 - \xi_1} \|P\|_{\xi_2} .
\end{aligned}$$

In the last line, d is the total degree of P. $\square$

Chapter 4

Moments of the β-ensemble and maps

4.1 Introduction

Since the seminal work of Brézin, Itzykson, Parisi and Zuber [Bré+78], random matrix techniques have been a powerful tool for enumerating maps. Informally, a map is a graph embedded in a compact surface. An important problem in combinatorics is to count the number of maps with some constraints: on the number of edges, the genus of the surface the graph is embedded in, the number and degrees of the faces or vertices. In the sequel the data of these degrees will be called the vertex or face *profile* respectively. It is a partition of twice the number of edges. This problem was studied first by Tutte, who gave several important results concerning the number of planar maps [Tut63; Tut68]. Random matrix theory allows to tackle this problem using tools from analysis and probability. It allowed to address questions concerning the moduli space of curves [HZ86] but also 2d quantum gravity [DGZ95].

Arguably the most famous random matrix models are the Gaussian orthogonal, unitary, and symplectic ensembles (GOE, GUE, and GSE respectively). These are respectively real symmetric, complex Hermitian, and quaternionic self-adjoint matrices whose coefficients are (up to symmetry) independent real, complex, or quaternionic Gaussian variables. Moments of the GUE are related to the enumeration of orientable maps, while moments of the GOE or the GSE are related to possibly non-orientable maps, see [Cic82] or [MW03].

Let X^N be a $N \times N$ matrix sampled according to one of these three matrix ensemble. Denote by $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ its eigenvalue. Up to a scaling factor, the eigenvalues $\boldsymbol{\lambda}$ are distributed according to

$$\mathrm{d}\nu_\beta^N(\boldsymbol{\lambda}) = \frac{1}{\mathcal{Z}_\beta^N} |\Delta(\boldsymbol{\lambda})|^\beta \, \mathrm{e}^{-\frac{\beta N}{4}\sum_{i=1}^N \lambda_i^2} \mathrm{d}\boldsymbol{\lambda}, \tag{4.1}$$

where $\mathcal{Z}_\beta^N$ is the partition function, $\Delta(\boldsymbol{\lambda}) = \prod_{1\le i<j\le N}(\lambda_i - \lambda_j)$ is the Vandermonde determinant, $\mathrm{d}\boldsymbol{\lambda} = \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N$ is the Lebesgue measure on $\mathbb{R}^N$, and $\beta > 0$ is one of $\beta = 1$, $\beta = 2$, or $\beta = 4$ if X^N is sampled according to the GOE, the GUE, or the GSE respectively.

The probability measure ν_β^N is the β-ensemble introduced in Section 2.3.2. It is a well-defined probability measure for all $\beta > 0$. We may ask whether the moments of the β-ensemble are related to the enumeration of maps when $\beta \notin \{1, 2, 4\}$.

In a series of articles Goulden and Jackson [GJ96] studied what they called the *map series*:

$$M_\alpha(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{z}) = 2\alpha z \frac{\partial}{\partial z} \ln \sum_\theta z^{|\theta|/2} \frac{J_\theta(\boldsymbol{y}, \alpha) J_\theta(\boldsymbol{x}, \alpha)}{\langle J_\theta, J_\theta \rangle_\alpha} \left[p_2(\boldsymbol{z})^{|\theta|/2} \right] J_\theta(\boldsymbol{z}, \alpha),$$

where the sum is on all partitions of integers θ (including the empty partition), J is a Jack polynomial, $\langle \cdot \rangle_\alpha$ is an inner product between symmetric polynomials, and $\left[p_2(\boldsymbol{z})^{|\theta|/2}\right] J_\theta(\boldsymbol{z}, \alpha)$ denotes the coefficient of $p_2^{|\theta|/2}(\boldsymbol{z})$ in the expansion of $J_\theta(\boldsymbol{z}, \alpha)$ in terms of power sum symmetric polynomials. The Jack polynomials, defined in [Jac70], are symmetric polynomials which constitute a continuous deformation between the Schur function at $\alpha = 1$, and the zonal polynomials at $\alpha = 2$.

It has been shown by Goulden, Jackson, and Harer [GHJ01] that the map series is related to the measure ν_β^N through the formal relation

$$M_{2/\beta}(\boldsymbol{p}(\boldsymbol{y}), N, z) = \beta z \frac{\partial}{\partial z} \ln \nu_\beta^N \left[\mathrm{e}^{\frac{\beta}{2} \sum_{k\geq 1} \frac{z^{k/2}}{k} p_k(y) p_k(\lambda)} \right],$$

where $\boldsymbol{p}(\boldsymbol{y}) = (p_k(\boldsymbol{y}))_{k\geq 1}$. Note that in general, this identity holds only as an equality between formal series: differentiating any number of times and taking the parameters to zero on both sides yield the same result.

The maps series can be expressed in the basis of symmetric polynomials as

$$M_{2/\beta}(\boldsymbol{p}(\boldsymbol{x}), \boldsymbol{p}(\boldsymbol{y}), z) = \sum_{n\geq 0} \sum_{\mu,\nu} c_{\mu,\nu,n}(2/\beta - 1) p_\mu(\boldsymbol{x}) p_\nu(\boldsymbol{y}) z^n,$$

where the coefficients $c_{\mu,\nu,n}(b)$ are related to number of maps with n edges and profile of vertices and faces specified by μ and ν. The maps enumerated are orientable when $b = 0$ and possibly non-orientable when $b = 1$. A more general series could be considered, the hypermap series where maps are replaced by bipartite maps and the profiles of the two types of vertices are specified independently. Goulden and Jackson conjectured [GJ96] that the coefficients of the hypermap series are polynomials in $b = \frac{2}{\beta} - 1$ encoding a measure of how non-orientable a map is. In our case, this becomes:

Conjecture 4.1.1 (Marginal b-conjecture)**.** *For all $n, f \geq 1$, $\nu \vdash 2n$,*

$$\sum_{\substack{\mu\vdash 2n \\ l(\mu)=f}} c_{\mu,\nu,n}(b) = \sum_{\mathfrak{m}} b^{\vartheta(\mathfrak{m})},$$

where the sum is on maps (possibly non-orientable) with n edges and profile of vertices and faces prescribed by μ and ν. The exponent $\vartheta(\mathfrak{m})$ is a measure of how non-orientable $\mathfrak{m}$ is, with $\vartheta(\mathfrak{m}) = 0$ if and only if $\mathfrak{m}$ is orientable.

This marginal b-conjecture has been solved by LaCroix [LaC09, Theorem 4.16]. He described the exponent ϑ using an inductive procedure, similar to Tutte's decomposition of maps. A generalization of the conjecture of Goulden and Jackson has been studied by Chapuy and Dołęga [CD22]. They studied a b-deformation of a tau function of the 2-Toda integrable hierarchy, and showed that it is a generating function of generalized branched covering of the sphere, with b-weight depending on a measure of non-orientability.

We study the cumulants of the β-ensemble, related to the marginal b-conjecture, and propose a different answer than the one of LaCroix. It based on the tridiagonal matrix model for the β-ensemble, introduced by Dumitriu and Edelman [DE02]. This tridiagonal ensemble was used by Abdesselam, Anderson, and Miller [AAM14] to recover the result that number of planar maps correspond to the leading order of the cumulants of the GUE ($\beta = 2$). A remarkable fact was that the natural combinatorial objects they obtain were *mobiles*, a family of labelled trees shown to be in bijection with rooted planar maps. Their proof relied on the Brydges-Kennedy-Abdesselam-Rivasseau formula, a complicated identity coming from cluster expansion theory. We simplify and generalize their work. We show that the cumulants of the symmetric power sum polynomials in the eigenvalues of the β-ensemble admit a large N expansion whose coefficients are expressed using *suitably labelled maps*, a family of maps with labelled vertices introduced by Bouttier, Fusy and Guitter [BFG14] which are in bijection with a family of maps generalizing the mobiles. For the two leading orders, we are able to reinterpret the coefficients as being sums of maps on the sphere or on the projective plane $\mathbb{RP}^2$ respectively. This is done using a novel many-to-one mapping that relate some suitably labelled maps and maps on $\mathbb{RP}^2$.

Another approach based on the Virasoro (or Dyson-Schwinger) equations is proposed by Cassia et al. [CPZ24]. However, they do not give combinatorial interpretations in terms of maps of the *generalized Catalan numbers* they obtain.

We prove the following Theorem in Section 4.4.1.

Theorem 4.1.2. *Let $\boldsymbol{n} = (n_1, \ldots, n_l) \in \mathbf{N}^l$ be a partition of an even integer $n \geq 2$ with l parts, i.e. $n_1 + \cdots + n_l = n$, and $\theta \in \mathcal{C}_{\boldsymbol{n}}$. Set for convenience $m = n/2 - l + 1$. We have the following expansion for the cumulants of the β-ensemble:*

$$\left(\frac{2}{\beta}\right)^{1-l} \frac{\kappa_l(\boldsymbol{n})}{N^{2-l}} = \sum_{v=0}^{m} \frac{1}{N^v} \sum_{u+q+r=v} \left(\frac{2}{\beta}\right)^u \frac{(-1)^q B_r}{m+1-v} \binom{r+m-v}{r} \langle e_q \rangle_{\theta, u+l-1}, \tag{4.2}$$

where $\langle \cdot \rangle_{\theta,p}$ denotes a sum over suitably labelled maps with face profile θ and $n/2-p$ vertices that are not local minima, introduced in Section 4.4.1. The sequence $(B_r)_{r\geq 0} = (1, -1/2, 1/6, \ldots)$ is the sequence of Bernoulli numbers, defined inductively by

$$\sum_{k=0}^{n} \binom{n+1}{k} (-1)^k B_k = \delta_{n,0} \text{ for all } n \geq 0. \tag{4.3}$$

We observe a mismatch between the expansion obtained by LaCroix, in terms of orientable and non-orientable maps, and the expansion of Theorem 4.1.2, in which the main combinatorial objects are orientable, vertex-labelled maps. We are able to understand the sub-leading order of our expansion through a novel many-to-one mapping described in Section 4.5. Given a permutation θ with $c(\theta)$ cycles, the construction described in Section 4.5 gives a $2^{c(\theta)-1}$-to-1 mapping between the set of pointed labelled maps on the projective plane with face determined by $\theta\bar{\theta}$, and the set of suitably labelled maps with two local minima and face profile θ. The precise result is stated in Theorem 4.5.42. Thanks to Theorem 4.5.42, the two leading orders of the expansion of Theorem 4.1.2 can be interpreted in the following way.

Corollary 4.1.3. *Let $\mathcal{M}_0(\theta)$ be the number of edge-labelled planar maps with face profile θ, and $\mathcal{M}_{1/2}(\theta)$ be the number of edge-labelled maps on $\mathbb{RP}^2$ with face profile θ. We have*

$$\kappa_l(\boldsymbol{n}) = N^{2-l} \left(\frac{2}{\beta}\right)^{l-1} \left(\#\mathcal{M}_0(\theta) + \frac{2^{1-l}}{N}(\frac{2}{\beta} - 1)\#\mathcal{M}_{1/2}(\theta) + \mathcal{O}\left(\frac{1}{N^2}\right)\right).$$

In Section 4.2, we give first expressions for the cumulants of power sums of the "eigenvalues" of the β-ensemble. We then describe in Section 4.3 the main combinatorial objects involved, labelled maps. We recall known facts and bijections, and give a combinatorial way to describe them in terms of Motzkin paths and permutations. We use these objects to re-express the cumulants of the beta ensemble in terms of sum of combinatorial objects in Section 4.4. Finally, in Section 4.5, we propose a novel many-to-one mapping that bridges the gap between our expansion and expansion in terms of non-orientable maps on the projective plane. This result, Theorem 4.5.42 is one of the main result of our article.

4.2 The moments of the β-ensemble

We now compute a formula for the moments of the β-ensemble, which we now define.

Definition 4.2.1. *Let $l \geq 1$ and $k_1, \ldots, k_l \geq 0$ be integers. The moment of order $\boldsymbol{k} = (k_1, \ldots, k_l)$ is*

$$m_l(\boldsymbol{k}) := \nu_\beta^N \left(p_{k_1}(\boldsymbol{\lambda}) \cdots p_{k_l}(\boldsymbol{\lambda})\right),$$

where p_k is the power sum symmetric polynomial

$$p_k(\boldsymbol{\lambda}) = \sum_{i=1}^{N} \lambda_i^k.$$

4.2.1 The tridiagonal model

For some time, it was an open question whether there was a matrix model for μ_β^N, that is, whether there existed a simple random matrix whose eigenvalues are distributed according to μ_β^N. The celebrated paper of Dumitriu and Edelman [DE02] gave a positive answer to this question by exhibiting a symmetric tridiagonal real random matrix with independent (up to symmetry) entries. Recall that the chi distribution with parameter $\alpha > 0$, χ_α, is the measure on $\mathbb{R}^+$ whose density with respect to the Lebesgue measure is

$$\rho_{\chi_\alpha}(x) = \frac{x^{\alpha-1}\mathrm{e}^{-x^2/2}}{2^{(\alpha/2)-1}\Gamma(\alpha/2)} \,. \tag{4.4}$$

Theorem 4.2.2 ([DE02]). *Let $\beta > 0$ be real and let $(a_i, b_j)_{1\le i\le N, 1\le j<N}$ be a family of real independent random variables such that for all $i \in [N]$, a_i is a standard Gaussian, and for all $i \in [N-1]$, $\sqrt{2}b_i$ is distributed according to the chi distribution with parameter $(N-i)\beta$. The eigenvalues of the random tridiagonal matrix*

$$T_\beta^N = \sqrt{\frac{2}{N\beta}}\begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 & \dots \\ b_1 & a_2 & b_2 & 0 & 0 & \dots \\ 0 & b_2 & a_3 & b_3 & \ddots & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & b_{N-2} & a_{N-1} & b_{N-1} \\ 0 & \cdots & 0 & 0 & b_{N-1} & a_N \end{pmatrix}, \tag{4.5}$$

are distributed according to μ_β^N, the β-ensemble distribution (4.1).

Remark 4.2.3. Notice that the factor $N\beta/2$ is not present in the result of Dumitiru and Edelman. Here, we use a different convention for the variance of the Gaussian entries: the eigenvalues are thus rescaled by a factor $\sqrt{\frac{N\beta}{2}}$.

4.2.2 Combinatorial interpretation of the χ distribution

The moments of the standard Gaussian distribution $\mu_k^{\mathcal{N}}$ are

$$\mu_k^{\mathcal{N}} = \begin{cases} (k-1)!! & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

They can be directly interpreted combinatorially as follows. Denote by $\mathfrak{S}(I)$ the group of permutation of the elements of a finite subset I of $\mathbf{N}$, and by Id its neutral element. For convenience, we write for $n \ge 1$, $[n] = \{1, 2, \dots, n\}$ and $\mathfrak{S}_n \coloneqq \mathfrak{S}([n])$.

Definition 4.2.4. *The set of matchings of a finite set $I \subset \mathbf{N}$ is*

$$\mathcal{I}^*(I) = \{\alpha \in \mathfrak{S}(I) \colon \alpha^2 = \mathrm{Id}, \forall i \in I, \alpha(i) \neq i\}\,,$$

i.e., the set of involution without fixed point. Let $n \ge 1$, we write $\mathcal{I}_n^* = \mathcal{I}^*([n])$.

The set of matchings could also be defined as the set of partitions of $[n]$ whose blocks are of size 2. The following lemma is well-known.

Lemma 4.2.5. *Let $n \ge 1$, we have*

$$\#\mathcal{I}_{2n}^* = (2n-1)!! = \mu_{2n}^{\mathcal{N}}\,.$$

In Lemma 4.2.6 below, we give a combinatorial interpretation of the moments of a chi variable in terms of permutations. Before stating it, we introduce more notation related to permutations. Fix a finite set $I \subset \mathbf{N}$. The group of permutations $\mathfrak{S}(I)$ acts naturally on I. Let $\sigma \in \mathfrak{S}(I)$. The orbits of the action of σ on I define a partition $\mathcal{O}(\langle\sigma\rangle)$ of I. Each block $B \in \mathcal{O}(\langle\sigma\rangle)$ defines a cyclic permutation $\sigma|_B \in \mathfrak{S}(B)$. We write

$$\operatorname{Cycles}(\sigma) = \{\sigma|_B \colon B \in \mathcal{O}(\langle\sigma\rangle)\}\,,$$

and set

$$\#\sigma = \#\operatorname{Cycles}(\sigma) = \#\mathcal{O}(\langle\sigma\rangle).$$

The support of σ is

$$\operatorname{Supp}\sigma = \{i \in I \colon \sigma(i) \neq i\}.$$

We denote by $|\sigma|$ the length of σ, i.e. the minimal number l such that σ can be written as a product of l transpositions. The length satisfies $|\sigma| = \#I - \#\sigma$.

Lemma 4.2.6. *Let $n \geq 1$ an integer, $\alpha > 0$, and X be random variable distributed as χ_α. We have*

$$\mathbb{E}\left[\left(\frac{X}{\sqrt{2}}\right)^{2n}\right] = \sum_{i=0}^{n-1}\left(\frac{\alpha}{2}\right)^{n-i}\#\{\sigma \in \mathfrak{S}_n \colon |\sigma| = i\} = \sum_{\sigma\in\mathfrak{S}_n}\left(\frac{\alpha}{2}\right)^{\#\sigma}.$$

Proof. We have

$$\mathbb{E}\left[\left(\frac{X}{\sqrt{2}}\right)^{2n}\right] = \frac{\Gamma(n+\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})} = \prod_{i=1}^{n}\left(\frac{\alpha}{2}+i-1\right),$$

where we used that for $x > 0$, $\Gamma(x+1) = x\Gamma(x)$. We expand the product to obtain

$$\mathbb{E}\left[\left(\frac{X}{\sqrt{2}}\right)^{2n}\right] = \sum_{i=0}^{n}\left(\frac{\alpha}{2}\right)^{n-i}\sum_{\substack{J\subset[n]\\ \#J=i}}\prod_{j\in J}(j-1).$$

The number of transpositions of the form $(i\,j)$ with $i < j$ is $j-1$. A product of strictly increasing transpositions is a product

$$\tau_r\tau_{r-1}\cdots\tau_1$$

of transpositions $\tau_i = (a_i\,b_i)$ such that $a_i < b_i$ for all i and $b_i < b_j$ for all $i < j$. A permutation of length i admits a unique decomposition as a product of i strictly increasing transpositions. Thus,

$$\sum_{\substack{J\subset[k]\\ \#J=i}}\prod_{j\in J}(j-1) = \#\{\sigma \in \mathfrak{S}_n \colon |\sigma| = i\}.$$

The second equality is a consequence of the fact that $|\sigma| = n - \#\sigma$. □

4.2.3 Moments and Motzkin paths

The computation of powers of a tridiagonal matrix naturally involves the notion of Motzkin paths.

Definition 4.2.7. *A Motzkin bridge of size $k \geq 1$ and with profile $\theta \in \mathfrak{S}_k$ is a function $\gamma \colon [k] \to \mathbf{N}$ such that for all $i \in [k]$,*

$$|\gamma(i) - \gamma(\theta(i))| \leq 1\,.$$

We define the two following sets of Motzkin bridges

$$\operatorname{Motz}_{k,0}(\theta) = \{\gamma \colon [k] \to \mathbf{N} \colon \min\gamma = 0, |\gamma(i) - \gamma(\theta(i))| \leq 1\},$$
$$\operatorname{Motz}_k^{[N]}(\theta) = \{\gamma \colon [k] \to [N] \colon |\gamma(i) - \gamma(\theta(i))| \leq 1\}.$$

We observe that for $k \geq 1$,

$$\mathrm{Tr}\left((T_\beta^N)^k\right) = \sum_{i_1,\dots,i_k=1}^{N} (T_\beta^N)_{i_1 i_2} \cdots (T_\beta^N)_{i_{k-1} i_k} (T_\beta^N)_{i_k i_1} = \sum_{\gamma \in \mathrm{Motz}_k^N((1\,2\cdots k))} \prod_{i=1}^{k} (T_\beta^N)_{\gamma(i)\gamma(i+1)},$$

since $(T_\beta^N)_{ij} = 0$ if $|i-j| > 1$. We use the convention that $\gamma(k+1) = \gamma(1)$. We proceed similarly for a product of such traces. For $k_1, \dots, k_l \geq 1$ and $k = \sum_{i=1}^{l} k_i$, define

$$\theta(\boldsymbol{k}) = (1 \cdots k_1) \cdots (\sum_{i=1}^{l-1} k_i + 1 \cdots \sum_{i=1}^{l} k_i). \tag{4.6}$$

We then have

$$\prod_{i=1}^{l} \mathrm{Tr}\left((T_\beta^N)^{k_i}\right) = \sum_{\gamma \in \mathrm{Motz}_k^N(\theta(\boldsymbol{k}))} \prod_{i=1}^{k} (T_\beta^N)_{\gamma(i)(\gamma\theta(\boldsymbol{k}))(i))}.$$

We are ready to compute the moments $m_l(\boldsymbol{k})$ (recall Definition 4.2.1).

Definition 4.2.8. *Let γ be a Motzkin bridge of size k and profile θ. We define for $\epsilon \in \{+1, 0, -1\}$, the set*

$$\Delta\gamma_\epsilon = \{i \in [k] \colon \gamma(\theta(i)) - \gamma(i) = \epsilon\}.$$

We write $\Delta\gamma_{+1} = \Delta\gamma_+$ and $\Delta\gamma_{-1} = \Delta\gamma_-$ for convenience.

A permutation $\sigma \in \mathfrak{S}_k$ is said to be compatible with γ if $\gamma \circ \sigma = \gamma$, and its restrictions to $\Delta\gamma_+, \Delta\gamma_-$, and $\Delta\gamma_0$ satisfy the following conditions:

- *$\sigma_- := \sigma|_{\Delta\gamma_-}$ is a permutation of $\Delta\gamma_-$,*
- *$\sigma_+ := \sigma|_{\Delta\gamma_+}$ is the identity on $\Delta\gamma_+$,*
- *$\sigma_0 := \sigma|_{\Delta\gamma_0}$ is a matching (recall Definition 4.2.4).*

The set of permutations compatible with γ is denoted by $\mathfrak{S}^\gamma$.

Note that no permutation can be compatible with γ if $\#\Delta\gamma_0$ is odd: in that case σ_0 cannot be a matching.

Proposition 4.2.9. *Let $l \geq 1$, $\boldsymbol{k} \in (\mathbf{N}^*)^l$, and $k = \sum_{i=1}^{l} k_i$. The moments can be expressed as*

$$m_l(\boldsymbol{k}) = (\frac{2}{N\beta})^{k/2} \sum_{\gamma \in \mathrm{Motz}_k^{[N]}(\theta(\boldsymbol{k}))} \sum_{\sigma \in \mathfrak{S}^\gamma} \left(\frac{\beta}{2}\right)^{\#\sigma_-} \prod_{\pi \in \mathrm{Cycles}(\sigma_-)} (\gamma(\pi) - 1), \tag{4.7}$$

where $\gamma(\pi)$ denotes the value of γ on the support of the cycle π.

Proof. We introduce the local times at height n and $n + 1/2$:

$$\begin{aligned} t_n &= \#L_n & &\text{with } L_n = \{i \in \Delta\gamma_0 \colon \gamma(i) = n\}, \\ t_{n+1/2} &= \#L_{n+1/2} & &\text{with } L_{n+1/2} = \{i \in \Delta\gamma_- \colon \gamma(i) = n+1\}. \end{aligned}$$

They allow us to write the moments as

$$m_l(\boldsymbol{k}) = \sum_{\gamma \in \mathrm{Motz}_k^{[N]}(\theta)} \mathbb{E}\left(\prod_{i=1}^{k} (T_\beta^N)_{\gamma(i)\gamma(\theta(i))}\right) = (\frac{2}{N\beta})^{k/2} \sum_{\gamma \in \mathrm{Motz}_k^N(\theta)} \mathbb{E}\left(\prod_{n=1}^{N} a_n^{t_n} b_n^{2t_{n+1/2}}\right).$$

with $\theta = \theta(\boldsymbol{k})$. Notice that there is a factor 2 in front of $t_{n+1/2}$ as we have to account for indices in $\Delta\gamma_-$. By independence and Lemma 4.2.6, we have

$$m_l(\boldsymbol{k}) = (\frac{2}{N\beta})^{k/2} \sum_{\gamma \in \mathrm{Motz}_k^{[N]}(\theta)} \prod_{n=1}^{N} \mathbb{E}\left(a_n^{t_n}\right) \mu_\beta^N \left(b_n^{2t_{n+1/2}}\right)$$
$$= (\frac{2}{N\beta})^{k/2} \sum_{\gamma \in \mathrm{Motz}_n^{[N]}(\theta)} \prod_{n=1}^{N} (\#\mathcal{I}^*(L_n)) \left(\sum_{\sigma \in \mathfrak{S}(L_{n+1/2})} \left(\frac{\beta}{2} n\right)^{\#\sigma} \right).$$

Notice that

$$\prod_{n=1}^{N} (\#\mathcal{I}^*(L_n)) \left(\sum_{\sigma \in \mathfrak{S}(L_{n+1/2})} \left(\frac{\beta}{2} n\right)^{\#\sigma} \right) = \sum_{\sigma \in \mathfrak{S}^\gamma} \left(\frac{\beta}{2}\right)^{\#\sigma_-} \prod_{n=1}^{N} n^{\#\sigma|_{L_{n+1/2}}}.$$

Indeed, $\#\mathcal{I}^*(L_n)$ is the number of matching on L_n (corresponding to the permutation σ_0 in Definition 4.2.8), and the condition that $\gamma\sigma = \gamma$ in Definition 4.2.8 corresponds to each cycle of $\sigma|_{\Delta\gamma_-}$ having support in one of the $L_{n+1/2}$.

Finally, we have for each $\sigma \in \mathfrak{S}^\gamma$ that

$$\prod_{n=1}^{N} n^{\#\sigma|_{L_{n+1/2}}} = \prod_{\pi \in \mathrm{Cycles}(\sigma_-)} (\gamma(\pi) - 1).$$

We obtain

$$m_l(\boldsymbol{k}) = (\frac{2}{N\beta})^{k/2} \sum_{\gamma \in \mathrm{Motz}_n^N(\theta)} \sum_{\sigma \in \mathfrak{S}^\gamma} \left(\frac{\beta}{2}\right)^{\#\sigma_-} \prod_{\pi \in \mathrm{Cycles}(\sigma_-)} (\gamma(\pi) - 1)\,.$$

□

It will prove more convenient to consider cumulants rather than moments, so as to have connected rather than disconnected objects. Let us first recall the definition of a cumulant.

Definition 4.2.10. *Let $X_1, \ldots, X_n$ be n real random variables. The joint cumulants $(\kappa_l)_{l \geq 1}$ of these random variables are l-multilinear symmetric maps defined inductively by*

$$\mathbb{E}\left[X_{i_1} \cdots X_{i_l}\right] = \sum_{\Pi \text{ partition of } [l]} \prod_{V \in \Pi} \kappa_{|V|}(X_{i_k}, k \in V)\,.$$

We denote by $\kappa_l(\boldsymbol{k})$ the joint cumulant of $p_{k_1}(\boldsymbol{\lambda}), p_{k_2}(\boldsymbol{\lambda}), \ldots, p_{k_l}(\boldsymbol{\lambda})$ under ν_β^N.

Proposition 4.2.9 then translates into the following result.

Corollary 4.2.11. *Let $l \geq 1$, $\boldsymbol{k} \in (\mathbf{N}^*)^l$, and $k = \sum_{i=1}^{l} k_i$. The cumulants can be expressed as*

$$\kappa_l(\boldsymbol{k}) = (\frac{2}{N\beta})^{k/2} \sum_{\gamma \in \mathrm{Motz}_k^{[N]}(\theta(\boldsymbol{k}))} \sum_{\substack{\sigma \in \mathfrak{S}^\gamma \\ \langle \theta(\boldsymbol{k}), \sigma \rangle \text{ acts transitively on } [k]}} \prod_{\pi \in \mathrm{Cycles}(\sigma_-)} \frac{\beta}{2} (\gamma(\pi) - 1)\,, \tag{4.8}$$

where $\langle \theta(\boldsymbol{k}), \sigma \rangle$ is the group generated by $\theta(\boldsymbol{k})$ and σ.

Proof. Let I be finite set and G be a subgroup of $\mathfrak{S}(I)$. We denote $\mathcal{O}(G)$ the set of orbits of the action of G on I. It is a partition of I. We decompose the formula of Proposition 4.2.9 depending on the number of orbits of $\langle\theta(\boldsymbol{k}),\sigma\rangle$ and get

$$\begin{aligned}
m_l(\boldsymbol{k}) &= (\frac{2}{N\beta})^{k/2} \sum_{\gamma\in\mathrm{Motz}_k^{[N]}(\theta(\boldsymbol{k}))} \sum_{\sigma\in\mathfrak{S}^\gamma} \prod_{\pi\in\mathrm{Cycles}(\sigma_-)} \frac{\beta}{2}\left(\gamma(\pi)-1\right) \\
&= \sum_{\Pi\in\mathcal{P}_l} (\frac{2}{N\beta})^{k/2} \sum_{\gamma\in\mathrm{Motz}_k^{[N]}(\theta(\boldsymbol{k}))} \sum_{\substack{\sigma\in\mathfrak{S}^\gamma \\ \mathcal{O}(\langle\theta(\boldsymbol{k}),\sigma\rangle)=\Pi}} \prod_{\pi\in\mathrm{Cycles}(\sigma_-)} \frac{\beta}{2}\left(\gamma(\pi)-1\right) \\
&= \sum_{\Pi\in\mathcal{P}_l} \prod_{B\in\Pi} \left[(\frac{2}{N\beta})^{k_B/2} \sum_{\gamma\in\mathrm{Motz}_{k_B}^{[N]}(\theta((k_i,i\in B)))} \sum_{\sigma} \prod_{\pi\in\mathrm{Cycles}(\sigma_-)} \frac{\beta}{2}\left(\gamma(\pi)-1\right) \right],
\end{aligned}$$

where the second sum is on permutations $\sigma\in\mathfrak{S}^\gamma$ such that $\langle\theta(\boldsymbol{k}),\sigma\rangle$ acts transitively on $[\#B]$. We introduced the notation $k_B=\sum_{i\in B}k_i$ for $B\subset[l]$.

On the other hand, the moments are related to the cumulant through

$$m_l(\boldsymbol{k}) = \sum_{\Pi}\prod_{B\in\Pi}\kappa_{|B|}(k_i, i\in B)\,.$$

This implies that $\kappa_l(\boldsymbol{k})$ coincides with the cumulant of $(p_{k_i})_{1\leq i\leq l}$ under ν_β^N. ☐

4.2.4 Large N expansion

We now consider the large N asymptotics of the moments and cumulants computed in Proposition 4.2.9 and Corollary 4.2.11. We prove the following large N expansion.

Proposition 4.2.12. *Let $l\geq 1$ and $\boldsymbol{n}=(n_1,\dots,n_l)\in(\mathbf{N}^*)^l$ with $n=\sum_{i=1}^l n_i$,*

$$\kappa_l(\boldsymbol{n}) = \sum_{p+q+r+s=n/2} \sum_{\substack{\gamma\in\mathrm{Motz}_{n,0}(\theta(\boldsymbol{n})) \\ \sigma\in\mathfrak{S}_\gamma,|\sigma|=p \\ \langle\theta(\boldsymbol{n}),\sigma\rangle \text{ transitive}}} \left(\frac{2}{\beta}\right)^p \frac{(-1)^q B_r}{s+1}\binom{r+s}{r} \times N^{s+1-n/2} e_q\left(\gamma(\pi);\pi\in\mathrm{Cycles}(\sigma_-)\right).$$

where $e_q(x_1,\dots,x_m)=\sum_{1\leq i_1<i_2<\dots<i_q\leq m}\prod_{j=1}^q x_{i_j}$ is the q-th elementary symmetric polynomial and the numbers $(B_r)_{r\geq 0}$ are the Bernoulli numbers defined in (4.3).

Notice that the permutation $\theta(\boldsymbol{n})$ has l cycles and thus if $|\sigma|=p<l-1$ the group $\langle\theta(\boldsymbol{n}),\sigma\rangle$ cannot act transitively on $[n]$. Thus, the leading order of $\kappa_l(\boldsymbol{n})$, obtained when s is maximal in the sum above under the constraint $\geq l-1$. The leading order is given by taking $p=l-1, q=0, r=0, s=n/2-l+1$, which gives

$$\kappa_l(\boldsymbol{n}) = \left(\frac{2}{\beta}\right)^{l-1} N^{-l+2} \frac{\#\left\{(\gamma,\sigma)\colon \begin{array}{c}\gamma\in\mathrm{Motz}_{n,0}(\theta(\boldsymbol{n})),\sigma\in\mathfrak{S}_\gamma, \\ \langle\theta(\boldsymbol{n}),\sigma\rangle \text{ acts transitively on } [n] \\ |\sigma|=l-1\end{array}\right\}}{n/2-l+2} + \mathcal{O}\Big(N^{-l+1}\Big).$$

We shall see in Section 4.4 a combinatorial description of the terms of the expansion.

Proof. We start by noticing that in (4.8), we can make the bijective change of variable

$$\begin{cases}\mathrm{Motz}_n^{[N]}(\theta(\boldsymbol{n})) & \to \{(h,\gamma')\in\mathbf{N}^*\times\mathrm{Motz}_{n,0}(\theta(\boldsymbol{n}))\colon h\geq\max\gamma'+1\} \\ \gamma & \mapsto (h,\gamma')=(\max\gamma,\max\gamma-\gamma).\end{cases}$$

We have $\Delta\gamma_+ = \Delta\gamma'_-$, $\Delta\gamma_- = \Delta\gamma'_+$, and $\Delta\gamma_0 = \Delta\gamma'_0$. We get

$$\kappa_l(\boldsymbol{n}) = (\frac{2}{N\beta})^{n/2} \sum_{\gamma\in \mathrm{Motz}_{n,0}(\theta(\boldsymbol{n}))} \sum_{h\geq \max\gamma+1} \sum_{\substack{\sigma\in\mathfrak{S}^{h-\gamma} \\ \langle\theta(\boldsymbol{n}),\sigma\rangle \text{ transitive}}} \prod_{\pi\in \mathrm{Cycles}(\sigma|_{\Delta\gamma_+})} \frac{\beta}{2}\left(h-1-\gamma(\pi)\right).$$

Given $\gamma \in \mathrm{Motz}_{n,0}(\theta(\boldsymbol{n}))$, we choose any bijection $\tilde{\phi}\colon \Delta\gamma_+ \to \Delta\gamma_-$ satisfying for all $i\in\Delta\gamma_+$: i and $\gamma(i)$ are part of the same cycle of $\theta(\boldsymbol{n})$ and

$$\gamma(\tilde{\phi}(i)) = \gamma(i)+1.$$

Such a bijection exists since γ is a Motzkin bridge: for any level k, there are as many up-steps between k and $k+1$ as down-steps between $k+1$ and k. We extends the definition of $\tilde{\phi}$ to a bijection (actually, an involution) $\phi\colon [n]\to[n]$ by

$$\phi(i) = \begin{cases} \tilde{\phi}(i) & \text{if } i\in\Delta\gamma_+ \\ \tilde{\phi}^{-1}(i) & \text{if } i\in\Delta\gamma_- \\ i & \text{if } i\in\Delta\gamma_0. \end{cases}$$

This bijection allows us to define the change of variable

$$\sigma\in\mathfrak{S}_{h-\gamma} \mapsto \phi^{-1}\circ\sigma\circ\phi \in \mathfrak{S}_\gamma.$$

We thus have

$$\begin{aligned} &\sum_{\substack{\sigma\in\mathfrak{S}_{h-\gamma} \\ \langle\theta(\boldsymbol{n}),\sigma\rangle \text{transitive}}} \prod_{\pi\in \mathrm{Cycles}(\sigma|_{\Delta\gamma_+})} \frac{\beta}{2}\left(h-1-\gamma(\pi)\right) \\ &\qquad = \sum_{\substack{\sigma\in\mathfrak{S}_{\gamma} \\ \langle\theta(\boldsymbol{n}),\sigma\rangle \text{ transitive}}} \prod_{\pi\in \mathrm{Cycles}(\phi\circ\sigma\circ\phi|_{\Delta\gamma_+})} \frac{\beta}{2}\left(h-1-\gamma(\pi)\right) \\ &\qquad = \sum_{\substack{\sigma\in\mathfrak{S}_{\gamma} \\ \langle\theta(\boldsymbol{n}),\sigma\rangle \text{ transitive}}} \prod_{\pi\in \mathrm{Cycles}(\sigma|_{\Delta\gamma_-})} \frac{\beta}{2}\left(h-\gamma(\pi)\right). \end{aligned}$$

Note that the transitivity condition is not changed because of our constraints that i and $\phi(i)$ must be part of the same cycle of $\theta(\boldsymbol{n})$.

The cumulant can be rewritten as

$$\kappa_l(\boldsymbol{n}) = (\frac{2}{N\beta})^{n/2} \sum_{\gamma\in \mathrm{Motz}_{n,0}(\theta(\boldsymbol{n}))} \sum_{h=\max\gamma+1}^{N} \sum_{\substack{\sigma\in\mathfrak{S}^{\gamma} \\ \langle\theta(\boldsymbol{n}),\sigma\rangle \text{ is transitive}}} \prod_{\pi\in \mathrm{Cycles}(\sigma_-)} \frac{\beta}{2}\left(h-\gamma(\pi)\right).$$

Notice that when $1\leq h\leq \max\gamma$, the product is 0 so that

$$\kappa_l(\boldsymbol{n}) = (\frac{2}{N\beta})^{n/2} \sum_{\gamma\in \mathrm{Motz}_{n,0}(\theta(\boldsymbol{n}))} \sum_{h=1}^{N} \sum_{\substack{\sigma\in\mathfrak{S}^{\gamma} \\ \langle\theta(\boldsymbol{n}),\sigma\rangle \text{ is transitive}}} \prod_{\pi\in \mathrm{Cycles}(\sigma_-)} \frac{\beta}{2}\left(h-\gamma(\pi)\right). \tag{4.9}$$

Set $p=|\sigma|$, and notice that $c(\sigma_-) = n/2-p$. We expand the product as

$$\prod_{\pi\in \mathrm{Cycles}(\sigma_-)} \frac{\beta}{2}\left(h-\gamma(\pi)\right) = \left(\frac{\beta}{2}\right)^{n/2-p} \sum_{q+u=n/2-p} (-1)^q e_q\left(\gamma(\pi);\pi\in \mathrm{Cycles}(\sigma_-)\right) h^u. \tag{4.10}$$

Recall Faulhaber's formula

$$\sum_{h=1}^{N} h^u = \sum_{r+s=u} \binom{r+s}{r} \frac{B_r}{s+1} N^{s+1} \,. \tag{4.11}$$

Substituting (4.10) and (4.11) in (4.9), we get

$$\kappa_l(\boldsymbol{n}) = \sum_{p+q+r+s=n/2} \sum_{\substack{\gamma \in \mathrm{Motz}_{n,0}(\theta_{\boldsymbol{n}}) \\ \sigma \in \mathfrak{S}_\gamma, |\sigma|=p \\ \langle \theta(\boldsymbol{n}), \sigma\rangle \text{ is transitive}}} \left(\frac{2}{\beta}\right)^p \frac{(-1)^q B_r}{s+1} \binom{r+s}{r}$$
$$\times N^{s+1-n/2} e_q\left(\gamma(\pi); \pi \in \mathrm{Cycles}(\sigma_-)\right),$$

as wanted. □

4.3 Maps and labelled hypermaps

We introduce the notions needed to reinterpret (4.8) in terms of maps. We first recall the definition of a map, and then discuss the bijection between suitably labelled maps and labelled hypermaps introduced by Bouttier, Fusy, and Guitter [BG14]. It is a generalization of the bijection between pointed planar maps and labelled trees called mobiles introduced in [BDG04]. In the process, we give a combinatorial description of these objects in terms of permutations and Motzkin paths.

4.3.1 Maps and permutations

We recall some notions pertaining to maps. For more details, see [LZ04] and [MT01].

Definition 4.3.1. *Let Γ be a graph (with possibly multi-edges and loops), seen as a 1-dimensional cell complex, and S be a connected compact surface without boundaries. A **cellular embedding** of Γ into S is an embedding ι of Γ into S, such that $S \setminus \iota(\Gamma)$ is a disjoint union of simply connected open sets of S. The corresponding **embedded graph** is the tuple (Γ, S, ι).*

Definition 4.3.2. *Two embedded graphs (Γ, S, ι) and (Γ', S', ι') are **isomorphic** if there exists an orientation-preserving homeomorphism $\varphi\colon S \to S'$ such that $\varphi \circ \iota(\Gamma) = \iota'(\Gamma')$ and $\iota'^{-1} \circ \varphi \circ \iota|_\Gamma$ is a graph isomorphism $\Gamma \to \Gamma'$. A **map** is a class of connected embedded graphs taken up to isomorphism.*

Remark 4.3.3. In the sequel, we will consider maps with additional structure that depends on the underlying graph Γ or the embedding ι. In these cases, the homeomorphism φ is taken to furthermore preserve this additional structure.

Definition 4.3.4. *A **hypermap** is a map whose vertices are colored in white or black, and such that every edge connects a white vertex to a black vertex. An **edge-labelled map** is a map whose edges are labelled in a bijective way from 1 to n, where n is the number of edges in the map.*

Until the end of the Section, we work exclusively with maps on orientable surfaces. We will consider possibly non-orientable maps in Section 4.5.

We orient each edge from its white vertex to its black vertex, and thus define a left and right side of the edge. Let e and f be respectively an edge and a face of a hypermap. We say that e is incident to f or that f is incident to e if f is at the left of e.

Finally, we will use the notion of corners and rooted maps.

Definition 4.3.5. *Consider a map $\mathfrak{m}$. A **corner** of $\mathfrak{m}$ is a vertex together with an angular sector comprised between two edges incident to the same vertex and the same face. A **rooted map** is a map with the choice of a distinguished oriented corner.*

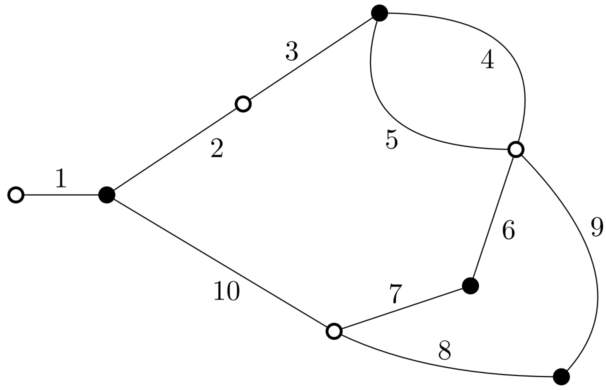


Figure 4.1: A hypermap with labelled edges.

In the orientable case, a result of Edmonds [Edm60] (see for instance [LZ04] for a modern account) shows that edge-labelled hypermaps with n edges are in bijection with pairs of permutations $(\theta, \sigma) \in \mathfrak{S}_n^2$. We now recall this construction.

Construction 4.3.6. *Consider a hypermap $\mathfrak{h}$. We construct a pair of permutations $(\theta_\mathfrak{h}, \sigma_\mathfrak{h})$. Each black vertex of $\mathfrak{h}$ corresponds to a cycle of θ and each white vertex to a cycle of σ. Let w be a white vertex. Assume that when going around w in the clockwise direction, we encounter the edges labelled $u_1, \ldots, u_k$. We associate to w the cycle $\rho = (u_1\, u_2\, \cdots\, u_k)$ in σ. We do this for all the white vertices of the map, and proceed similarly for the black vertices, which corresponds to cycles of θ.*

This construction defines an injective function from the set of bipartite labelled maps with n edges to $\mathfrak{S}_n^2$. This map can be shown to be surjective (see [Edm60]).

It is convenient to define the permutation $\varphi_\mathfrak{h} = \theta_\mathfrak{h}^{-1}\sigma_\mathfrak{h}^{-1}$. Each cycle of $\varphi_\mathfrak{h}$ corresponds to a face of $\mathfrak{h}$. Assume a face f of $\mathfrak{h}$ is incident to edges labelled $u_1, \ldots, u_k$, and that these labels are encountered in that order when going around the boundary of the face in clockwise order. Then, $(u_1\, \ldots\, u_k)$ is a cycle of $\varphi_\mathfrak{h}$. This result is proved in [LZ04, Proposition 1.3.16].

Example 4.3.7. The hypermap $\mathfrak{h}$ depicted in Figure 4.1 is encoded by the permutations

$$\begin{aligned}
\theta_\mathfrak{h} &= (1\,2\,10)(3\,4\,5)(6\,7)(8\,9)\\
\sigma_\mathfrak{h} &= (1)(2\,3)(4\,9\,6\,5)(7\,8\,10)\\
\varphi_\mathfrak{h} &= (1\,10\,9\,3)(2\,5\,7)(4)(6\,8).
\end{aligned}$$

Remark 4.3.8. In particular, the hypermap has $c(\varphi_\mathfrak{h})$ faces. This number of faces is related to the genus $g_\mathfrak{h}$ of $\mathfrak{h}$ according to Euler's formula:

$$(c(\theta_\mathfrak{h}) + c(\sigma_\mathfrak{h})) - n + c(\varphi_\mathfrak{h}) = 2 - 2g_\mathfrak{h}. \tag{4.12}$$

Remark 4.3.9. Maps $\mathfrak{m}$ with non-colored vertices can be seen as hypermaps by coloring the vertices of $\mathfrak{m}$ black and adding a white vertex in the middle of each edge. We obtain a hypermap $\mathfrak{h}(\mathfrak{m})$ with all its white vertices of degree 2. A hypermap obtained in such a way can have its edge labelled and be described by a pair $(\theta_{\mathfrak{h}(\mathfrak{m})}, \sigma_{\mathfrak{h}(\mathfrak{m})})$ with $\sigma_\mathfrak{h}$ a matching (recall Definition 4.2.4).

We call the edges of $\mathfrak{h}(\mathfrak{m})$ the *half-edges* of $\mathfrak{m}$. To each half-edge h, we denote the vertex to which it is attached by $\mathrm{vert}(h)$. Furthermore, there is a unique distinct half-edge h' such that h and h' form an edge. We say that h' is the **counterpart** of h. We will say that we label the half-edges of $\mathfrak{m}$ to mean that we label the edges of $\mathfrak{h}(\mathfrak{m})$. We can then set

$$\theta_\mathfrak{m} := \theta_{\mathfrak{h}(\mathfrak{m})}, \quad \sigma_\mathfrak{m} := \sigma_{\mathfrak{h}(\mathfrak{m})}, \text{ and } \varphi_\mathfrak{m} := \varphi_{\mathfrak{h}(\mathfrak{m})}.$$

Remark 4.3.10. Let I be a finite subset of $\mathbf{N}^*$, $\mathfrak{h}$ be a hypermap, and $E_\mathfrak{h}$ be the set of edges of the hypermap. When a hypermap $\mathfrak{h}$ is edge-labelled with labels in I, it is equipped with a bijection $\lambda\colon E_\mathfrak{h} \to I$. For any permutation $\pi \in \mathfrak{S}(I)$, we can construct naturally the permutation of the edges

$$\pi^\lambda = \lambda^{-1} \circ \pi \circ \lambda \in \mathfrak{S}(E_\mathfrak{h}).$$

In particular, we define naturally the permutations of the edges $\theta_\mathfrak{h}^\lambda$, $\sigma_\mathfrak{h}^\lambda$, and $\varphi_\mathfrak{h}^\lambda$. We abuse notation in the sequel and omit the superscript λ when it is not ambiguous.

This construction also applies to maps: in this case we replace $\mathfrak{h}$ by a maps $\mathfrak{m}$, the set $E_\mathfrak{h}$ by the set $H_\mathfrak{m}$ of half-edges of $\mathfrak{m}$, and $\lambda\colon E_\mathfrak{h} \to I$ by a bijection $H_\mathfrak{m} \to I$.

4.3.2 Well-labelled hypermaps and suitably labelled maps

In [BDG04], Bouttier, Di Francesco, and Guitter introduced a celebrated bijection between maps and a family of trees with labelled vertices called *mobiles*. In the investigation of the 2-point functions, Bouttier, Fusy, and Guitter [BFG14] introduced a generalization of the bijection. It allowed them to relate *suitably labelled maps* of any genus and *well-labelled hypermaps*. We are going to use this bijection in the sequel.

Definition 4.3.11. *A suitably labelled map is a map $\mathfrak{m}$ such that each vertex v of $\mathfrak{m}$ carries a label $l(v) \in \mathbf{N}$ satisfying:*

- $\min_v l(v) = 0$,
- *for each edge e between vertices v and w we have $|l(v) - l(w)| \leq 1$.*

An edge between two vertices v and v' in a suitably labelled map is said to be frustrated if $l(v) = l(v')$. We denote by $\mathcal{S}_n$ the set of suitably labelled maps with n half-edges, and $\hat{\mathcal{S}}_n$ the set of such suitably labelled maps with no frustrated edges.

Note that the definition given here is slightly different from the one in [BFG14], where the authors allowed $l(v) \in \mathbf{Z}$ and no constraint on the minimum of the labels. We similarly give a modified version of a well-labelled hypermap that mirrors these changes.

Definition 4.3.12. *A well-labelled hypermap is a hypermap $\mathfrak{h}$ such that each white vertex w carries a label $l(w) \in \mathbf{N}^*$ satisfying:*

- $\min_w l(w) = 1$,
- *Let b be a black vertex and w, w' be two white vertices adjacent to b, with (b, w) and (b, w') consecutive edges, in that order in the clockwise orientation. Then, $l(w') \geq l(w) - 1$.*

Furthermore, if a white vertex w of degree 2 that is connected to black vertices b_1 and b_2 is preceded (in the clockwise direction) around both of b_1 and b_2 by white vertices w_1 and w_2 of label $l(w_1), l(w_2) \leq l(w)$, then it may be marked. We call these marked vertices frustrated vertices. *We denote by $\mathcal{H}_n$ the set of well-labelled hypermaps with n edges, and $\hat{\mathcal{H}}_n$ the set of such maps with no frustrated vertices.*

Note that the definition of well-labelled hypermap is given in [BFG14] in terms of face-bicolored Eulerian map. The definition we give, in terms of star-representation of a hypermap, is equivalent.

The cw-type of a face We now recall a notation introduced in [BFG14] to describe the faces of suitably labelled maps, and black vertices of hypermaps (corresponding in to dark faces of hypermaps in the conventions of [BFG14]).

Definition 4.3.13. *The cw-type of a face f of a suitably labelled map is the cyclic list of the labels of the vertices adjacent to f.*

The cw-type τ of a black vertex b of a well-labelled hypermap is the cyclic list of the the labels of the white vertices adjacent to b, in clockwise order.

The lower completion of τ, denoted by $c^\downarrow(\tau)$, is the cyclic list obtained by inserting $i-1, \ldots, j-1$ between two consecutive elements $i \leq j$ of τ.

From well-labelled hypermap to suitably labelled maps Let us recall briefly the construction of [BFG14] to go from a well-labelled hypermap to a suitably labelled map.

Construction 4.3.14. *Start from the well-labelled hypermap* $(\hat{\mathfrak{h}}, l)$. *Denote by* $\min f$ *the minimum label of a white vertex incident to a face* f. *For each face* f *we proceed as follows.*

1. *Add a new white vertex* w_f *labelled by* $\min f - 1$ *in the interior of* f.

2. *For each corner* c *incident to* f *and at a white vertex* w, *we attach a half-edge* h_c.

3. *For each added half-edge* h_c *attached to a white vertex* w *we consider the label* $l(w)$. *If* $l(w) = \min f$, *we attach this half-edge* h_c *to* w_f. *Otherwise, we connect* h_c *to the next corner in the counterclockwise order which is at a vertex labelled by* $l(w) - 1$. *We call this second corner the* successor *of* c.

We do this for all faces of $\hat{\mathfrak{h}}$. *Finally, we remove all the edges and the black vertices of* $\hat{\mathfrak{h}}$. *We obtain a map* $\hat{\mathfrak{m}}$.

The inverse construction is as follows.

Definition 4.3.15. *Let* h *be a half-edge in a suitably labelled map* $\hat{\mathfrak{m}}$, *incident to a face* f. *Let* h' *be the counterpart of* h. *Let* v *be the vertex incident to* h *and* v' *be the vertex incident to* h'. *We say that* h *is a* ***decreasing half-edge along*** f *if*

$$l(v') = l(v) - 1.$$

We say h *is an* ***increasing half-edge along*** f *if*

$$l(v') = l(v) + 1.$$

Construction 4.3.16. *Start from a suitably labelled map* $(\hat{\mathfrak{m}}, \ell)$.

1. *Color all the vertices of* $\hat{\mathfrak{m}}$ *white.*

2. *For each face* f *in* $\hat{\mathfrak{m}}$, *add a new black vertex* b *in its interior. For each decreasing half-edge incident to* f *and connected to a white vertex* w, *add an edge between* w *and* b.

3. *Erase all the edges of the original map, and the isolated white vertices.*

Theorem 4.3.17. *[BFG14, Theorem 1] Constructions 4.3.14 and 4.3.16 give a bijection between* $\hat{\mathcal{S}}_n$ *and* $\hat{\mathcal{H}}_n$. *For a well-labelled hypermap* $\hat{\mathfrak{h}}$ *corresponding to a suitably labelled map* $\hat{\mathfrak{m}}$,

- *each white vertex* w *of* $\hat{\mathfrak{h}}$ *corresponds to a non local minimum vertex* v *of* $\hat{\mathfrak{m}}$ *of the same label;*
- *this vertex* w *is a local maximum if and only if* v *is a local maximum in* $\hat{\mathfrak{m}}$;
- *each face of* $\hat{\mathfrak{h}}$ *corresponds to a local minimum vertex of* $\hat{\mathfrak{m}}$, *of label* $\min f - 1$;
- *each black vertex of* $\hat{\mathfrak{h}}$ *of cw-type* τ *corresponds to a face of* $\hat{\mathfrak{m}}$ *of cw-type* $c^{\downarrow}(\tau)$.

Actually, [BFG14, Theorem 1] is stated only as a bijection between $\hat{\mathcal{H}}_n$ and $\hat{\mathcal{S}}_n$. Generalizing this to the general case is straightforward using the duplication of edges trick, used in particular in [BDG04]. Given a well-labelled hypermap $(\mathfrak{h}, l)$ with frustrated vertices, we produce a suitably labelled map $(\hat{\mathfrak{m}}, \ell)$ with some of its white vertices marked. Indeed, the non-local minimum vertices of the map $\mathfrak{m}$ constructed in Construction 4.3.14 are the white vertices of $\mathfrak{h}$: the possible marking of white vertices in $\mathfrak{h}$ induce a marking of vertices in $\mathfrak{m}$. We call those marked vertices the frustrated vertices of $\hat{\mathfrak{m}}$.

Lemma 4.3.18. *The frustrated vertices of* $\hat{\mathfrak{m}}$ *are of degree* 2.

For each frustrated vertex v in $\hat{\mathfrak{m}}$ incident to two edges e_1 and e_2, we remove v from $\hat{\mathfrak{m}}$ and glue e_1 and e_2 together. We obtain a suitably labelled map $(\mathfrak{m}, \ell)$ with one frustrated edge for each removed frustrated vertex.

The construction works in the converse direction: consider a suitably labelled map $(\mathfrak{m}, \ell)$. For each frustrated edge between vertices v_1 and v_2, we add a new vertex v in the middle of it which we label by $\ell(v) = \ell(v_1) + 1 = \ell(v_2) + 1$. We obtain in this way a suitably labelled map without frustrated edge. The well-labelled hypermap produced by Construction 4.3.16 has naturally frustrated vertices. Consider such a vertex w, corresponding to a frustrated vertex v in $\mathfrak{m}$. The choice of labelling for v ensures that w satisfies the condition to be a frustrated vertex.

In remains to prove Lemma 4.3.18.

Proof of Lemma 4.3.18. Let v be a frustrated vertex in $\mathfrak{m}$, coming from a frustrated vertex in $\mathfrak{h}$. The vertex v is at least of degree 2: since w is of degree 2, it has two corners and thus at step 2 two half-edges get connected to it. For the degree to be at least 3, there must be another white vertex w' incident to the same face f as w, with $l(w') = l(w) + 1$. However, when going around f in the counterclockwise orientation, the label between two consecutive white vertices decrease at most by 1. this means that w' is the white vertex preceding w when going around f in the counterclockwise orientation. By definition of a marked vertex, we would have $l(w') \leq l(w)$, a contradiction. □

4.3.3 Encoding the labelled hypermaps

In Section 4.2.4, we expressed the cumulants of the β-ensemble in terms of a sum over a Motzkin path $\gamma \in \mathrm{Motz}_{n,0}(\theta)$ and a permutation. $\sigma \in \mathfrak{S}^{\gamma}$. In this Section, we explain how this data allow us to define a labelled hypermap, and thus a suitably labelled map by the Bouttier-Fusy-Guitter construction 4.3.14. To introduce the main result of this Section, we define the notion of restriction of a permutation.

Definition 4.3.19. *Let $I \subset I'$ two finite sets and $\pi \in \mathfrak{S}(I')$. We define the jump in π with respect to J by*

$$J_{\theta,I}(j) = \min\{p \in \mathbf{N}^* \colon \pi^p(j) \in I\} \text{ for all } j \in I.$$

We define the restriction of π to J by

$$\pi|_{J\to I}(j) = \pi^{J_{\theta,I}(j)}(j) \text{ for } j \in I.$$

.

Note that in general $\pi|_{J\to J} \in \mathfrak{S}(J)$ differs from $\pi|_J \colon J \to \pi(J)$.

In the following Proposition, we make use of the following abuse of notation. Let $(\mathfrak{h}, l)$ be a well-labelled hypermap, whose edges are labelled by $I \subset \mathbf{N}^*$. For each $i \in I$, there is a unique white vertex w_i incident to the edge labelled i. We write

$$l(i) = l(w_i).$$

The main result of this Section is the following.

Proposition 4.3.20. *Let $n \in \mathbf{N}^*$, $\theta \in \mathfrak{S}_n$, and $I \subset [n]$. For all $\pi \in \mathfrak{S}(I)$ we define*

$$\mathcal{H}(\theta) := \left\{ (\hat{\mathfrak{h}}, l) \in \mathcal{H}_n \colon \exists I \subset [n], \quad \begin{array}{l} \bullet\, \hat{\mathfrak{h}} \text{ is edge-labelled by } I \\ \bullet\, \theta_{\hat{\mathfrak{h}}} = \theta_{I\to I} \\ \bullet\, l \circ \theta_{\hat{\mathfrak{h}}} = l + J_{\theta,I} - 2 \end{array} \right\}.$$

and

$$\mathfrak{C}(\theta) = \left\{ (\gamma, \sigma) \in \mathrm{Motz}_{n,0}(\theta) \times \mathfrak{S}_n \colon \quad \begin{array}{l} \bullet\, \sigma \in \mathfrak{S}_\gamma \\ \bullet\, \langle \theta(\boldsymbol{n}), \sigma \rangle \text{ acts transitively on } [n] \end{array} \right\}.$$

Construction 4.3.21 below gives a bijection

$$\mathfrak{C}(\theta) \to \mathcal{H}(\theta).$$

The set $\mathfrak{C}(\theta, I)$ appear naturally in the expression of the cumulants. The last condition is a technical assumption needed for a proper labelling of the edges of the corresponding hypermap.

Construction 4.3.21. *The inverse of Construction 4.3.6 defines a edge-labelled hypermap $\mathfrak{h}$. Notice that each cycle of σ_+ – corresponding to an element of $\Delta\gamma_+$ – corresponds to a white vertex of degree 1 in $\mathfrak{h}$. We remove these vertices to obtain a new hypermap $\hat{\mathfrak{h}}$. The frustrated vertices of $\hat{\mathfrak{h}}$ are the vertices corresponding to the cycles of σ_0. We have*

$$\theta_{\hat{\mathfrak{h}}} = \theta|_{\Delta\gamma_- \sqcup \Delta\gamma_0 \to \Delta\gamma_- \sqcup \Delta\gamma_0} \text{ and } \sigma_{\hat{\mathfrak{h}}} = \sigma|_{\Delta\gamma_- \sqcup \Delta\gamma_0}.$$

We now explain how γ induces a labelling of the white vertices of $\hat{\mathfrak{h}}$. Let w be a white vertex of $\hat{\mathfrak{h}}$ that corresponds to $\pi \in \mathrm{Cycles}(\sigma_-) \cup \mathrm{Cycles}(\sigma_0)$. We set

$$l(w) = \begin{cases} \gamma(\pi) & \text{if } \pi \in \mathrm{Cycles}(\sigma_-) \\ \gamma(\pi) + 1 & \text{if } \pi \in \mathrm{Cycles}(\sigma_0). \end{cases}$$

We write in the sequel $(\hat{\mathfrak{h}}, l) = \Psi(\gamma, \theta, \sigma)$.

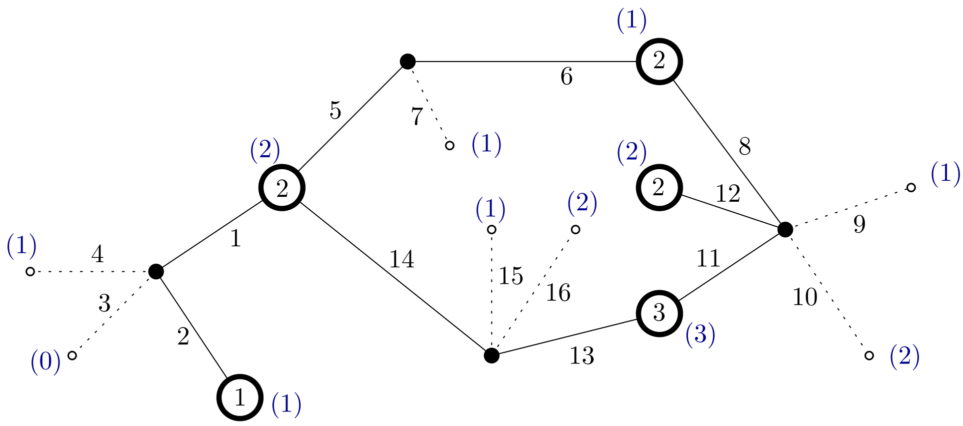


Figure 4.2: A labelled hypermap $(\hat{\mathfrak{h}}, l) = \Psi(\gamma, \theta, \sigma)$. We wrote in parenthesis the value of the path γ at each vertex. We displayed in dotted edges the edges to white vertices corresponding to elements of $\Delta\gamma_+$. They belong to $\mathfrak{h}$ but not to $\hat{\mathfrak{h}}$.

Example 4.3.22. The labelled hypermap displayed in Figure 4.2 is obtained from

$$\begin{aligned} (\gamma(i))_{i\in[16]} &= (2,1,0,1,2,1,1,1,1,2,3,2,3,2,1,2) \\ \theta &= (1\,2\,3\,4)(5\,6\,7)(8\,9\,10\,11\,12)(13\,14\,15\,16) \\ \sigma &= (1\,5\,14)(6\,8)(11\,13). \end{aligned}$$

Note that

$$\begin{aligned} \Delta\gamma_+ &= \{3,4,7,9,10,15,16\} \\ \Delta\gamma_0 &= \{6,8\} \\ \Delta\gamma_- &= \{1,2,5,11,12,13,14\}. \end{aligned}$$

Proof of Proposition 4.3.20. We start by showing that $(\hat{\mathfrak{h}}, l) = \Psi(\gamma, \theta, \sigma)$ is a well-labelled hypermap. Recall that $\min \gamma = 0$. Thus, $\min_w l(w) \geq 1$, and for i such that $\gamma(i) = 0$ we have $\gamma(\theta(i)) \in \{0, 1\}$ so that i is the label of an edge connected to a white vertex w. The label of w is $l(w) = 1$. We have shown that $\min_w l(w) = 1$.

Consider a black vertex b, and two white vertices w and w' such that (b, w) and (b, w') are consecutive edges around b, in that order in the clockwise direction. Let i, j be the labels of these edges. We have $i, j \in \Delta\gamma_0 \cup \Delta\gamma_-$. Thus, as γ is a Motzkin path, we have that $\gamma(i) \leq \gamma(j) + 1$. It implies that $l(w) - 1 \leq l(w')$. The hypermap is thus well-labelled.

We remark that by construction its edges are labelled by elements of $I = \Delta\gamma_- \sqcup \Delta\gamma_0$, that $\theta_{\hat{\mathfrak{h}}} = \theta|_{I \to I}$, and that since γ is a Motzkin path, the difference of label between two consecutive white vertices incident to a black vertex is given by the number of removed univalent white vertices minus one. It gives

$$\gamma_{\hat{\mathfrak{h}}} \circ \theta_{\hat{\mathfrak{h}}} = \gamma_{\hat{\mathfrak{h}}} + J_{\theta,I} - 2.$$

We now show that the mapping is a bijection by constructing the inverse map. Let $(\hat{\mathfrak{h}}, l) \in \mathcal{H}(\theta)$, and let I be a edge-labelling set. We will define a function $\tilde{l}$ on $[n]$. On I, it is defined by $\tilde{l}|_I = l$. We construct a new hypermap $\mathfrak{h}$ from $\hat{\mathfrak{h}}$. We add degree one white vertices, whose incident edges are labelled by elements of $[n] \setminus I$. Fix a black vertex b and let $\pi_b = (u_1 \ \ldots \ u_k)$ be the corresponding cycle in $\theta_{\hat{\mathfrak{h}}} = \theta_{I \to I}$. For each $j \in [k]$, let p_j be the jump as in Definition 4.3.19:

$$p_j = J_{\theta,I}(u_j) = \min \left\{ p \in \mathbf{N}^* : \theta^p(u_j) \in I \right\}.$$

We add after the edge labelled u_j in the clockwise orientation $p_j - 1 = J_{\theta,I}(u_j)$ edges connected to univalent white vertices, labelled by

$$\theta^1(u_j), \theta^2(u_j), \ldots, \theta^{p_j - 1}(u_j).$$

We set for all $1 \leq i \leq p_j - 1$,

$$\tilde{l}(\theta^i(u_j)) = \tilde{l}(u_j) - 2 + i.$$

Since we have (with the convention $u_{k+1} = u_1$)

$$l \circ \tilde{\theta}^{p_j}(u_j) = l(u_{j+1}) = l \circ \theta|_{I \to I}(u_j) = \tilde{l}(u_j) + p_j - 2,$$

we see that $\tilde{l}$ is a Motzkin path without flat steps. Let $F \subset I$ be the set of labels of edges connected to frustrated vertices. We then set for all $i \in [n]$

$$\gamma(i) = \tilde{l} - \mathbb{1}_F,$$

where $\mathbb{1}_F$ denote the indicator function of F. The function γ is a Motzkin path with $\Delta\gamma_0 = F$. We extend $\sigma_{\mathfrak{h}}$ by the identity to a permutation of $[n]$. By construction, we then have $\sigma \in \mathfrak{S}_\gamma$, and the connectedness of $\mathfrak{h}$ ensures that $\langle \theta, \sigma \rangle$ acts transitively on $[n]$. We thus have that

$$(\gamma, \sigma) \in \mathfrak{C}(\theta).$$

The mapping just constructed is the required inverse: it is clear that the mapping between (γ, σ) and $(\mathfrak{h}, \ell)$ is 1-to-1, and the inverse just define allow to reconstruct the data erased when going from $\mathfrak{h}$ to $\hat{\mathfrak{h}}$. □

Remark 4.3.23. In the case of the hypermap $(\hat{\mathfrak{h}}, l) = \Phi(\gamma, \theta, \sigma)$ with $\Delta\gamma_0 = \emptyset$, the clockwise cyclic type of a black vertex b corresponding to a cycle $\pi \in \mathrm{Cycles}(\theta)$ is the cyclic list with entries

$$\tau = \left(\gamma\pi^p(j)\right)_{p \in I_\pi}, \text{ with } I_\pi = \{1 \leq p \leq \#\,\mathrm{Supp}\,\pi : \pi^p(j) \in \Delta\gamma_-\}, \tag{4.13}$$

for $j \in \mathrm{Supp}\,\pi$. Otherwise said, it is the sublist of $(\gamma\pi^p(j))_{1 \leq p \leq \#\,\mathrm{Supp}\,\pi}$ obtained by keeping only the down steps in the original Motzkin bridge. The lower completion of τ is the cyclic list with entries

$$c^{\downarrow}(\tau) = (\gamma\pi^p(j))_{1 \leq p \leq \#\,\mathrm{Supp}\,\pi}, \text{ for some } j \in \mathrm{Supp}\,\pi\,.$$

4.3.4 Labelling the half-edges in the Bouttier-Fusy-Guitter bijection

The bijection of Theorem 4.3.17 is between sets of hypermap and maps that are not half-edge or edge-labelled. We now explain how the edge labels of a well-labelled hypermap $(\mathfrak{h}, l)$ get transported to a half-edge-labelling of a suitably labelled map $(\mathfrak{m}, \ell)$.

Fix a well-labelled hypermap $(\mathfrak{h}, l)$. In Construction 4.3.14, we add one edge for each corner of $\hat{\mathfrak{h}}$. We now explain how to label the two half-edges making up each of those edges, see Figure 4.3.

In step 3 of Construction 4.3.14 applied to $\hat{\mathfrak{h}}$, we added a half-edge h_c for each corner c in $\hat{\mathfrak{h}}$. The corner c is based at a white vertex w, and is delimited by two edges: e_1 to e_2 in the clockwise direction around w. Let i be the label of e_2. In $\hat{\mathfrak{m}}$, let h'_c be the next half-edge around w after h_c in the clockwise orientation . We label h'_c by i

With the previous procedure, we labelled only half of the total number of half-edges in $\hat{\mathfrak{m}}$: exactly the half-edges following the decreasing half-edges around white vertices.

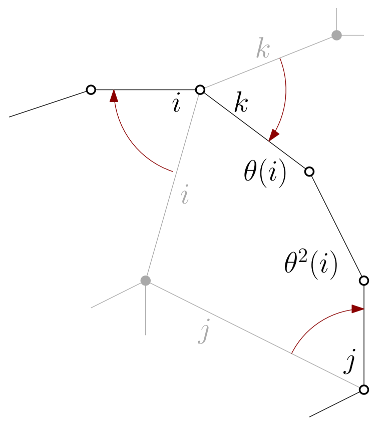


Figure 4.3: The labelling procedure. The vertices and edges of $\hat{\mathfrak{m}}$ are in black and the ones of $\hat{\mathfrak{h}}$ are in grey.

To label the other half-edges we proceed as follows. Let h be an increasing half-edge along a face f in $\hat{\mathfrak{m}}$. Assume that it is not the counterpart of an half-edge incident to a frustrated vertex. We explore the face f in the clockwise direction starting from h, and stop once we encounter a decreasing half-edge h'. Let $h_1 = h, h_2, \dots, h_k$ be the increasing half-edge we encounter during this exploration. Let i be the label of h'. We label h by $\theta^k(i)$.

Note that we do not label the counterparts of half-edges incident to frustrated vertices in the resulting suitable map with frustrated vertices. To finish the procedure, we remove the frustrated half-edges and construct a map $\mathfrak{m}$ from $\hat{\mathfrak{m}}$.

Construction 4.3.24. *Consider a frustrated vertex v in $\hat{\mathfrak{m}}$. Let h_1, h_2 be the two half-edges attached to v, and $\tilde{h}_1, \tilde{h}_2$ be two half-edges such that h_i and $\tilde{h}_i$ are counterparts of one another, for $i = 1, 2$.*

Assume that i_1 and i_2 are respectively the labels of h_1 and h_2. We remove v, h_1, and h_2. We connect $\tilde{h}_1$ and $\tilde{h}_2$ together. Finally, label $\tilde{h}_1$ by i_2 and $\tilde{h}_2$ by i_1. We denote the map we obtain after treating all frustrated vertices in this way by $\mathfrak{m}$.

The inverse construction is then as follows.

Construction 4.3.25. *Consider a suitably labelled map $(\mathfrak{m}, \ell)$. For each frustrated edge between vertices v_1 and v_2, made of half-edges $\tilde{h}_1$ and $\tilde{h}_2$ (with $\tilde{h}_1, \tilde{h}_2$ respectively attached to v_1, v_2), we proceed as follows. We add a vertex v with 2 half-edges h_1, h_2 attached to it. We label v by $\ell(v_1) + 1$. Assume that i_1, i_2 are the labels of $\tilde{h}_1, \tilde{h}_2$. We erase the labels of $\tilde{h}_1$ and $\tilde{h}_2$, and we label h_1 by i_2 and h_2 by i_1. The resulting labelled map, $(\hat{\mathfrak{m}}, \ell)$, is a suitably labelled map with no frustrated edge.*

Lemma 4.3.26. *This labelling is well-defined and no two half-edges receive the same label. Furthermore, $\varphi_{\mathfrak{m}} = \theta$.*

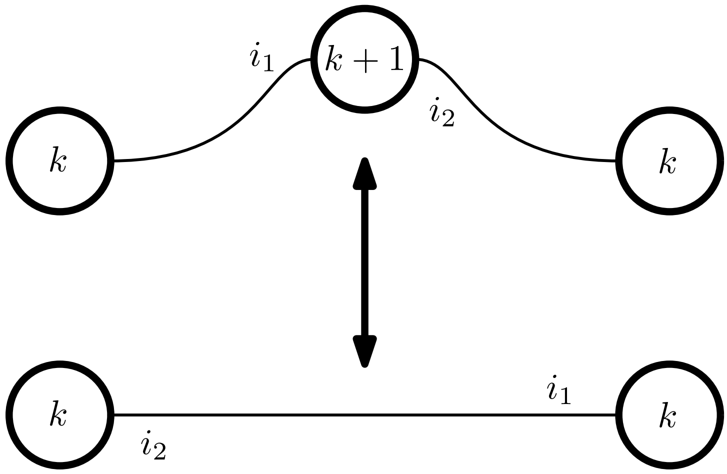


Figure 4.4: Constructions 4.3.24 and 4.3.25.

Proof. Each half-edge in $\hat{\mathfrak{m}}$ is either decreasing or increasing. If it is decreasing, it corresponds to a unique corner of a white vertex (and hence a unique edge) in $\hat{\mathfrak{h}}$. We see this in Construction 4.3.16. If it is increasing, it is incident to a unique face in $\hat{\mathfrak{m}}$ and the well-defined labelling of decreasing half-edges determines the labelling of the increasing half-edges.

Let us show that $\varphi_{\hat{\mathfrak{m}}} = \theta$. This will imply in particular that no two edges receive the same label. Let f be a face of $\hat{\mathfrak{m}}$ and h, h' be two half-edges incident to f, consecutive when going around the face in the clockwise direction. Let i and j be their labels: $j = \varphi(i)$. If h' is increasing, we immediately see that we have $j = \theta(i)$. If h' is decreasing, let h'' be the first decreasing half-edge along f encountered when exploring f in the counterclockwise direction starting from h. Let j' be its label. Considering Construction 4.3.16, we see that $\theta^p(j') = j$ for some $p \geq 1$ (for $i \in [p-1]$, $\theta^i(j')$ is the label of an increasing half-edge). We thus have $\theta^{p-1}(j') = i$ and $j = \theta(i)$. □

Using this labelling, we can describe the vertices of $\mathfrak{m}$ in terms of permutations. Let v be a white vertex in $\mathfrak{m}$. Let $u_1, \dots, u_d$ be the labels of the half-edges at v which are part of an edge connecting v to a vertex of strictly smaller label, encountered in this order when going in the clockwise direction around v. Define

$$\pi_v = (u_1 \, \dots \, u_d). \tag{4.14}$$

Let e be a frustrated edge in $\mathfrak{m}$ made of half-edges labelled by i and j. Define

$$\pi_e = (i\, j).$$

Lemma 4.3.27. *Let e be a frustrated edge in $\mathfrak{m}$, w be the corresponding frustrated white vertex in $\hat{\mathfrak{h}}$, and π be the cycle corresponding to w. Then, $\pi_e = \pi$.*

Let v be a vertex in $\mathfrak{m}$, w be the corresponding white vertex in $\hat{\mathfrak{h}}$, and π be the cycle corresponding to w. Then, $\pi_v = \pi$.

In particular, if w is of degree d, then v has exactly d neighbors of label $l(v) - 1$.

Proof. A frustrated edge e in $\mathfrak{m}$ is constructed from a frustrated vertex v in $\hat{\mathfrak{m}}$ of degree 2. This latter vertex is constructed from a frustrated vertex w in $\hat{\mathfrak{h}}$. If w is represented by $(i\, j)$, the two labels of the incident half-edges get transported by the labelling procedure: the two half-edges connected to v are decreasing half-edges. This shows the first claim.

Let w be a white vertex in $\hat{\mathfrak{h}}$, $\pi = (u_1 \, \dots \, u_k)$ be the corresponding cycle in σ_-, and v be the corresponding vertex in $\hat{\mathfrak{m}}$. Consider the half-edges added at step 2 of Construction 4.3.14. If w is of degree d, d such half-edges are attached to the corners of w. These half-edges are labelled by $u_1, \dots, u_k$ in that order when going around the vertex in the clockwise direction. At step 3, these half-edge are connected to vertices with label $l(v) - 1$. Now, consider a half-edge h' attached at step 2 to another white vertex w'. If at step 3, h' gets connected to w, then $l(w') = l(w) + 1$. It may be that w' is a frustrated vertex, and gets removed in Construction 4.3.24: then through the half-edge h', w is connected to a vertex of label $l(w)$. Hence, through this procedure we created exactly d edges connecting w to a vertex of degree $l(w) - 1 = l(v) - 1$, and the half-edges connected to v that are part of these d edges are labelled by $u_1, \dots, u_k$ in that order. □

Theorem 4.3.28. *Fix $n \geq 1$ and $\theta \in \mathfrak{S}_n$. Define*

$$\mathcal{S}(\theta) = \{(\mathfrak{m}, \ell) \in \mathcal{S}_n : \mathfrak{m} \textit{ is half-edge-labelled by } [n], \varphi_{\mathfrak{m}} = \theta\}.$$

The previous construction gives a bijection

$$\Psi \colon \mathfrak{C}(\theta) \to \mathcal{S}(\theta).$$

Furthermore, if $(\mathfrak{m}, \ell) = \Psi(\gamma, \sigma)$,

1. *each vertex v of $\mathfrak{m}$ that is not a local minimum corresponds to a cycle π_v as defined by (4.14), with $\pi_v \in \mathrm{Cycles}(\sigma_-)$, and has label $\ell(v) = \gamma(\pi_v)$;*
2. *each frustrated edge e of $\mathfrak{m}$ corresponds to a cycle $\pi_e \in \mathrm{Cycles}\,\sigma_0$ of length 2;*
3. *$\sigma = \prod_v \pi_v \prod_e \pi_e$ where the products are on the vertices that are not local minima and on the frustrated edges.*

Proof. If we forget about the labelling of the half-edges, the map Ψ is obtained by composing the bijection of Proposition 4.3.20 and the one of Theorem 4.3.17.

Fix $(\gamma, \sigma) \in \mathfrak{C}(\theta)$, $(\hat{\mathfrak{h}}, l) = \Phi(\gamma, \sigma)$, and $(\mathfrak{m}, \ell) = \Psi(\gamma, \sigma)$. The labelling of the edges of $\hat{\mathfrak{h}}$ determines the labelling of the decreasing half-edges of $\mathfrak{m}$. The unique determination of the other labels follows from the constraint that $\varphi_{\mathfrak{m}} = \theta$. Lemma 4.3.26 shows that with out choice of labelling we do have $\varphi_{\mathfrak{m}} = \theta$. Conversely, from a suitable map $(\mathfrak{m}, \ell)$ with $\varphi_{\mathfrak{m}} = \theta$, we can recover the labels of the corresponding hypermap by erasing the labels of the increasing half-edges.

The second part of the Theorem is a consequence of Lemma 4.3.27, and in the case of point 2 of Constructions 4.3.24 and 4.3.25 as well. □

4.4 Combinatorial description of the cumulants

We now re-express the cumulants (4.8) in terms of suitably labelled maps.

4.4.1 Expression in terms of the distances

The cumulants can be rewritten in terms of sums over suitably labelled maps. Indeed, Theorem 4.3.28 allow us to replace the sum on Motzkin paths and permutation in Proposition 4.2.12 with a sum on suitably labelled maps. This will prove Theorem 4.1.2.

Before rewriting the expansion of cumulants in terms of suitably labelled maps, we reinterpret the terms $e_q\left(\gamma(\pi); \pi \in \mathrm{Cycles}(\sigma_-)\right)$. We now show these terms correspond to product of distances in a map. Consider a suitably labelled map $\mathfrak{m}$, and denote by $V_{\mathfrak{m}}^{\min}$ the set of local minima of $\mathfrak{m}$ and $V_{\mathfrak{m}}^{\star} = V_{\mathfrak{m}} \setminus V_{\mathfrak{m}}^{\min}$. For a vertex $v \in V_{\mathfrak{m}}$, we set

$$d_v = \min_{v^* \in V_{\mathfrak{m}}^{\min}} \left(d(v^*, v) + \ell(v^*)\right),$$

where $d(u, u')$ is the graph distance between two vertices u and u'. By the second part of Theorem 4.3.28, if $v \in V_{\mathfrak{m} \setminus V_{\mathfrak{m}}^{\min}}$ corresponds to a cycle π, then the label of v corresponds to $\gamma(\pi)$. On the other hand, the label of v is d_v, as explained in [BFG14, Remark 1]. The argument goes as follow: consider any geodesic from v to some $v^* \in V_{\mathfrak{m}}^{\min}$, the labels along the geodesic are necessarily weakly decreasing (by steps of 0 or 1). There exists a choice of v^* and of a geodesic with strictly decreasing labels to v^*. In that case the length of the geodesic is the distance between v and v^* but also the difference of the labels of v and v^*.

Hence, using Theorem 4.3.28, we can rewrite the sum in Proposition 4.2.12 as

$$\sum_{\substack{\gamma\in\mathrm{Motz}_{n,0}(\theta_{\boldsymbol{n}})\\ \sigma\in\mathfrak{S}_\gamma,|\sigma|=p\\ \langle\theta(\boldsymbol{n}),\sigma\rangle \text{ is transitive}}} e_q\left(\gamma(x); c\in \mathrm{Cycles}(\sigma_-)\right) = \sum_{\substack{\mathfrak{m}\in\mathcal{S}(\theta)\\ \#V^\star_{\mathfrak{m}}=n/2-p}} e_q\left(d_v; v\in V^\star_{\mathfrak{m}}\right).$$

We introduce the notation of average over sum of maps of a symmetric polynomial f to be

$$\langle f\rangle_{\theta,p} = \sum_{\substack{\mathfrak{m}\in\mathcal{S}(\theta)\\ \#V^\star_{\mathfrak{m}}=n/2-p}} f\left(d_v; v\in V^\star_{\mathfrak{m}}\right).$$

This allows us to rewrite the expression for the cumulants in compact form:

$$\kappa_l(\boldsymbol{n}) = \sum_{p+q+r+s=n/2} \left(\frac{2}{\beta}\right)^p \frac{(-1)^q B_r}{s+1}\binom{r+s}{r} N^{s+1-n/2}\langle e_q\rangle_{\theta,p}. \tag{4.15}$$

This proves Theorem 4.1.2.

Remark 4.4.1. Notice that by Euler's formula:

$$2-2g_{\mathfrak{m}} = \left(\frac{n}{2}-p+\#V^{\min}_{\mathfrak{m}}\right)-\frac{n}{2}+l = l+\#V^{\min}_{\mathfrak{m}}-p.$$

Hence, when p increases, either the number of minima increase, or the genus increases.

4.4.2 Analysis of the first two orders

We now turn to the first two orders of the cumulants computed in Proposition 4.2.12. This Section will prove part of Corollary 4.1.3.

The leading order is obtained by considering the term $s=n/2-l+1, p=l-1, q=r=0$ in (4.15). It gives

$$\kappa_l(\boldsymbol{n}) = \left(\frac{2}{\beta}\right)^{l-1} N^{-l+2}\frac{\#\left\{\mathfrak{m}\in\mathcal{S}(\theta(\boldsymbol{n}))\colon \#V^\star_{\mathfrak{m}}=n/2-l+1\right\}}{n/2-l+2}(1+\mathcal{O}(1/N)).$$

Notice that by Remark 4.4.1, we are considering maps with

$$2-2g_{\mathfrak{m}} = 1+\#V^{\min}_{\mathfrak{m}}.$$

Since $\#V^{\min}_{\mathfrak{m}}\geq 1$, this equation is only satisfied when $g_{\mathfrak{m}}=0$ and $\#V^{\min}_{\mathfrak{m}}=1$. In this case, suitably labelled maps correspond exactly to pointed planar maps with face profile $\theta(\boldsymbol{n})$. As $n/2-l+2$ is the total number of vertices in the map, we get that

$$\kappa_l(\boldsymbol{n}) = \left(\frac{2}{\beta}\right)^{l-1} N^{-l+2}\#\{\text{edge-labelled planar maps with face profile } \theta(\boldsymbol{n})\}(1+\mathcal{O}(1/N)),$$

which is the first order of Corollary 4.1.3. We have recovered for all $\beta>0$ the result of Abdesselam, Anderson, and Miller [AAM14].

To treat the sub-leading order, we prove the following Proposition.

Proposition 4.4.2. *Let $n \in \mathbf{N}^*$ and $\theta \in \mathfrak{S}_n$. We define the set of suitably labelled map with two local minima:*

$$\mathcal{S}_2(\theta) = \left\{(\mathfrak{m}, \ell) \in \mathcal{S}(\theta) \colon \#V_{\mathfrak{m}}^{\min} = 2\right\}.$$

We have

$$N^{l-2}\kappa_l(\boldsymbol{n}) = \left(\frac{2}{\beta}\right)^{l-1} \#\mathcal{M}_0(\theta(\boldsymbol{n})) + \left(\frac{2}{\beta}\right)^{l-1} \frac{1}{N}\left(\frac{2}{\beta}-1\right)\frac{\#\mathcal{S}_2(\theta(\boldsymbol{n}))}{n/2-l+1} + \mathcal{O}\left(\frac{1}{N^2}\right),$$

with $\mathcal{M}_0(\theta(\boldsymbol{n}))$ the set of half-edge-labelled planar maps with face profile $\theta(\boldsymbol{n})$.

The sub-leading order of the cumulant is thus described by the suitably labelled maps on the sphere with exactly two local minima. In Section 4.5, we give another description of this object in terms of non-orientable maps on $\mathbb{RP}^2$. The full proof of Corollary 4.1.3 will follow from Proposition 4.4.2 and Theorem 4.5.42 proved in Section 4.5.

The proof is based on two mappings that we now introduce. We define the set of suitably labelled maps with two local minima and $m+1$ global minima

$$\mathcal{S}_2(\theta, m) = \left\{(\mathfrak{m}, \ell) \in \mathcal{S}(\theta) \colon \#V_{\mathfrak{m}}^{\min} = 2, \#\ell^{-1}(0) = m+1\right\}.$$

The integer m is in $\{0,1\}$. It is 0 if we consider maps with exactly one global minimum (vertex with label 0), and 1. We define the sets of suitably labelled maps with one local minimum and a choice of vertex with an additional label, which is either strictly positive, or zero:

$$\mathcal{S}_{1,+}(\theta) = \left\{(\mathfrak{m}, \ell, v, k) \colon (\mathfrak{m}, \ell) \in \mathcal{S}(\theta), v \in V_{\mathfrak{m}}^{\star}, \#V_{\mathfrak{m}}^{\min} = 1, 1 \le k < d(v, V_{\mathfrak{m}}^{\min})\right\}$$
$$\mathcal{S}_{1,0}(\theta) = \left\{(\mathfrak{m}, \ell, v) \colon (\mathfrak{m}, \ell) \in \mathcal{S}(\theta), v \in V_{\mathfrak{m}}^{\star}, \#V_{\mathfrak{m}}^{\min} = 1\right\}.$$

We construct a bijection $\phi_1 \colon \mathcal{S}_{1,+}(\theta) \to \mathcal{S}_2(\theta, 0)$ and a two-to-one mapping $\phi_2 \colon \mathcal{S}_{1,0}(\theta) \to \mathcal{S}_2(\theta, 1)$. The mappings are constructed by changing the label of the vertex with additional label to make it a local minimum. The bijection is as follows. Let $(\mathfrak{m}, \ell, v, k) \in \mathcal{S}_{1,+}(\theta)$. Denote the unique local minimum of $(\mathfrak{m}, \ell)$ by v^*. We define the labelling function by $\ell'(v) = k$, $\ell'(v^*) = 0$, and by

$$\ell'(u) = \min_{u^* \in \{v^*, v\}} \left(\ell'(u^*) + d(u^*, u)\right)$$

for any other vertex u. The second mapping ϕ_2 is constructed similarly, by replacing k with 0.

Lemma 4.4.3. *The labelled map $(\mathfrak{m}, \ell') \coloneqq \phi_1(\mathfrak{m}, \ell, v, k)$ is in $\mathcal{S}_2(\theta, 0)$ and ϕ_1 is a bijection.*
The labelled map $(\mathfrak{m}, \ell') \coloneqq \phi_2(\mathfrak{m}, \ell, v)$ is in $\mathcal{S}_2(\theta, 1)$ and ϕ_2 is two-to-one.

Proof. We first show that the map $(\mathfrak{m}, \ell')$ is suitably labelled. Let u, u' be two adjacent vertices in $\mathfrak{m}$. Let u_* (resp. u'_*) be the vertex in $\{v^*, v\}$ closest to u (resp. u'). Assume that $\ell'(u) \ge 2 + \ell'(u')$. As u and u' are adjacent, this means that $u_* \ne u'_*$. If $u_* = v$, then we have

$$d(u, v^*) \ge d(u, v) + k \ge 2 + d(u', v^*),$$

which contradicts the triangular inequality as $d(u, u') = 1$. If $u'_* = v$, we similarly have

$$d(u, v) + k \ge d(u, v^*) \ge 2 + d(u', v) + k,$$

a contradiction.

Two minima of $(\mathfrak{m}, \ell')$ are then v^* and v as any other vertex u adjacent to v has label

$$\ell'(u) = \min(d(v^*, u), k + d(v, u)) \ge \min(d(v^*, v) - 1, k) \ge k.$$

They are the only minima. Indeed, for all vertex $u \notin \{v, v^*\}$, let u^* be the vertex in $\{v, v^*\}$ closest to u. Let $u_0 = u, u_1, \ldots, u_{k-1}, u_k = u^*$ be the vertices on a geodesic from u to u^*. We then have

$$\ell'(u_1) \le d(u^*, u_1) = d(u^*, u) - 1.$$

This proves the first claim for ϕ_1 and ϕ_2.

Finally, we can invert ϕ_1 as follows: given $(\mathfrak{m}, \ell) \in \mathcal{S}_2(\theta, 0)$ and v^* the unique vertex with $\ell(v^*) = 0$ and v' the other minimum, we can construct a new labelling function

$$\ell'(v) = d(v^*, v).$$

This gives an element $(\mathfrak{m}, \ell', v', \ell(v')) \in \mathcal{S}_{1,+}(\theta)$.

For ϕ_2, given an element $(\mathfrak{m}, \ell) \in \mathcal{S}_2(\theta, 1)$, there are two vertices with label 0. We can thus construct two distinct preimages. □

Proof of Proposition 4.4.2. Lemma 4.4.3 implies

$$\#S_2(\theta, 0) = \#S_{1,+}(\theta) \text{ and } \#S_2(\theta, 1) = \frac{1}{2}\#S_{1,0}(\theta).$$

The sub-leading order of $\kappa_l(\boldsymbol{n})$ is then

$$\begin{aligned}
&\sum_{u+q+r=1} \left(\frac{2}{\beta}\right)^{l-1+u} \frac{(-1)^q B_r}{n/2-l+1} \binom{r+n/2-l}{r} N^{-l+1} \langle e_q \rangle_{\theta, l-1+u} \\
&= \left(\frac{2}{\beta}\right)^{l-1} \frac{N^{-l+1}}{n/2-l+1} \left(\frac{2}{\beta} \langle 1 \rangle_{\theta,l} - \langle e_1 \rangle_{\theta,l-1} + \frac{n/2-l+1}{2} \langle 1 \rangle_{\theta,l-1}\right) \\
&= \left(\frac{2}{\beta}\right)^{l-1} \frac{N^{-l+1}}{n/2-l+1} \left(\frac{2}{\beta} - 1\right) (\#\mathcal{S}_2(\theta, 0) + \#\mathcal{S}_2(\theta, 1)),
\end{aligned}$$

as

$$\begin{aligned}
\langle 1 \rangle_{\theta,l} &= \#\mathcal{S}_2(\theta, 0) + \#\mathcal{S}_2(\theta, 1) \\
\langle e_1 \rangle_{\theta,l-1} &= \#\mathcal{S}_{1,0}(\theta) + \#\mathcal{S}_{1,+}(\theta) \\
(n/2-l-1) \langle 1 \rangle_{\theta,l-1} &= \#\mathcal{S}_{1,0}(\theta).
\end{aligned}$$

□

We may wonder if a similar proof holds beyond the first sub-leading order. In theses cases, the mappings ϕ_1 and ϕ_2 must be defined differently.

4.5 A many-to-one map between suitably labelled maps, and non-orientable maps on $\mathbb{RP}^2$

We now propose a way to interpret the suitably labelled maps appearing in the sub-leading order of the expansion of the cumulants of the β-ensemble as non-orientable maps on $\mathbb{RP}^2$. To do so, we will interpret them as determining the lift of a map on a non-orientable surface on its orientable double-covering. Note that we produce a many-to-one mapping and not a bijection as we consider labelled non-orientable maps on $\mathbb{RP}^2$. Fixing a face profile using a permutation determines an orientation of the faces in the non-orientable map, an information that is redundant in an orientable map.

4.5.1 The orientation double covering

We recall a few facts on the orientable double covering of a non-orientable surface. See for instance the book of Lee [Lee12] for more.

Consider a connected manifold M. We can construct an orientable manifold $\hat{M}$ and a continuous surjective map $\pi\colon \hat{M} \to M$ such that $(\hat{M}, \pi)$ is a double covering of M. Informally, the construction is as follows: there are two choices of orientation locally around each point P of M. These two choices determine two sheets of the covering above a neighborhood of P. A surface is orientable if and only if we can make a consistent global choice of orientation. In that case, there are exactly two choices of global orientation, and $\hat{M}$ is the union of two disconnected copies of M: each copy corresponds to a choice of orientation. The manifold $\hat{M}$ is equipped with an involution without fixed point, which inverses the two sheets above P, or equivalently change the orientation around P.

This double covering is called the **orientation covering** of M. The connectedness property alluded to above is summarized in the following Theorem.

Theorem 4.5.1 ([Lee12, Theorem 15.41])**.** *Let $\pi\colon \hat{M}' \to M$ be the orientation covering of M. If M is a orientable, then $\hat{M}$ has two connected components and the restriction of π to any of these component is a homeomorphism. If M is not orientable, then $\hat{M}$ is connected.*

The orientation covering is unique is the sense of the following Theorem.

Theorem 4.5.2 (See for instance [Lee12, Theorem 15.42])**.** *Let $\pi'\colon \hat{M}' \to M$ be an orientable double covering of a non-orientable manifold M. Then, this covering is isomorphic to the orientation covering.*

In the sequel, we consider the orientation covering of $\mathbb{RP}^2$. It is topologically a sphere.

An important part of the mapping described in this Section is that a non-orientable map canonically defines a map on its orientation covering. Let us now detail why it is so. Note that starting from now, and until the end of the Section, the notation $\hat{\cdot}$ (as in $\hat{S}, \hat{\mathfrak{m}}, \ldots$) denote objects related to some orientation covering.

Construction 4.5.3. *Let $\mathfrak{m}$ be a non-orientable map. Consider a graph embedding (Γ, S, ι) in the class $\mathfrak{m}$. The surface S has a connected orientation covering $p\colon \hat{S} \to S$. We lift to $\hat{S}$ the image of Γ, $\iota(\Gamma)$. We obtain a graph embedding $(\hat{\Gamma}, \hat{S}, \hat{\iota})$ in $\hat{S}$ with twice the number of vertices, edges, and faces of $\mathfrak{m}$. We thus define a map $\hat{\mathfrak{m}}$ on the orientation covering of S. This map is well defined and does not depend on the particular choice of graph embedding (Γ, S, ι), since the orientation covering is unique up to isomorphism.*

The orientation covering $\hat{S}$ is equipped with an orientation-reversing involution without fixed point, $\mathrm{inv}_{\hat{S}}$. This involution descends to an involution on the set of vertices, edges and faces of (Γ, S, ι).

4.5.2 Combinatorial description of non-orientable maps

We now describe a way to encode non-orientable maps as triples of matchings (recall Definition 4.2.4). The construction we now describe is due to Tutte [Tut84] (see also [GR01]). Starting from now, and until the end of this Section, we abuse notation and define permutational models that acts either on sets of labels or set of half-edges or flags, as explained in Remark 4.3.10.

Definition 4.5.4. *Let $\mathfrak{m}$ be a map, orientable or non-orientable. Let h be an edge in $\mathfrak{m}$. We can distinguish between two sides of h. We call a side of a half-edge a **flag**. We denote by $\mathrm{Fl}_{\mathfrak{m}}$ the set of flags of $\mathfrak{m}$. Let $\mathfrak{f}$ be a flag on a half-edge h. There is a unique face f on this side $\mathfrak{f}$ of h. We say that f is incident to $\mathfrak{f}$.*

Let $\mathfrak{m}$ be a map with n half-edges. To define a flag-labelling function, we consider an extended set of labels I of size $2n$. A flag-labelling function is then a bijection $\lambda\colon \mathrm{Fl}_{\mathfrak{m}} \to I$. Let λ be such a function.

We then define three matchings $\tau_{\mathfrak{m}}, \rho_{\mathfrak{m}}, \mu_{\mathfrak{m}}$ as follows. We define their action on the set of flags, but by Remark 4.3.10, using the labelling λ, they equivalently act on the index set I. The cycles of $\tau_{\mathfrak{m}}$ are $(\mathfrak{f}\,\mathfrak{f}')$ where $\mathfrak{f}$ and $\mathfrak{f}'$ are the two flags associated to a same half-edge. Consider an edge e. Let $(\mathfrak{f}_1, \mathfrak{f}_1')$ and $(\mathfrak{f}_2, \mathfrak{f}_2')$ be two pairs of flags with $\mathfrak{f}_i, \mathfrak{f}_i'$ associated to the same half-edge of e, and $\mathfrak{f}_1$ and $\mathfrak{f}_2$ (resp. $\mathfrak{f}_1'$ and $\mathfrak{f}_2'$) on the same side of e. Then $(\mathfrak{f}_1\,\mathfrak{f}_2)(\mathfrak{f}_1'\,\mathfrak{f}_2')$ are two cycles of $\rho_{\mathfrak{m}}$. Finally, consider a corner of $\mathfrak{m}$. This corner is made of two flags, $\mathfrak{f}_1$ and $\mathfrak{f}_2$. Then, $(\mathfrak{f}_1\,\mathfrak{f}_2)$ is a cycle of $\mu_{\mathfrak{m}}$.

In the sequel, we will often use the following an extended set of labels. Let $n \in \mathbf{N}^*$. We define the set of "barred" integers $\{\bar{1}, \bar{2}, \ldots, \bar{n}\} = [\bar{n}]$. The extended set of label is denoted by $[n, \bar{n}] = \{1, \bar{1}, \ldots, n, \bar{n}\}$. For a subset $I \subset [n, \bar{n}]$, we define

$$\bar{I} = \{\bar{i}\colon i \in I \cap [n]\} \cup \{i\colon \bar{i} \in I \cap [\bar{n}]\}\,.$$

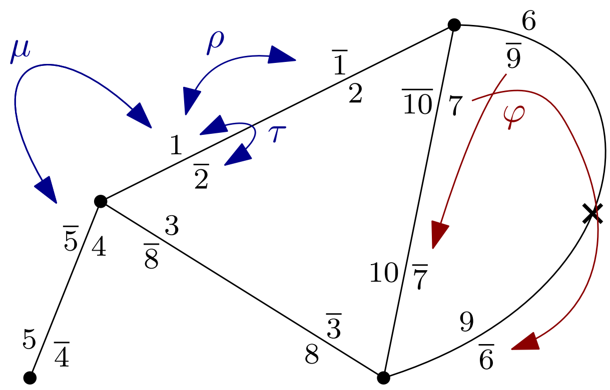


Figure 4.5: A non-orientable map with labelled flags. The edge with flags labelled $6, \bar{6}, 9, \bar{9}$ must be twisted to allow for an embedding in the plane. The blue and red arrows show the action of μ, ρ, τ and φ.

Example 4.5.5. The matchings describing the map displayed in Figure 4.5 are:

$$\begin{aligned}
\tau_{\mathfrak{m}} &= (1\,\bar{2})(2\,\bar{1})(3\,\bar{8})(8\,\bar{3})(4\,\bar{5})(5\,\bar{4})(6\,\bar{9})(9\,\bar{6})(7\,\overline{10})(10\,\bar{7})\\
\rho_{\mathfrak{m}} &= (1\,\bar{1})(2\,\bar{2})(3\,\bar{3})(4\,\bar{4})(5\,\bar{5})(6\,9)(7\,\bar{7})(8\,\bar{8})(\bar{6}\,\bar{9})(10\,\overline{10})\\
\mu_{\mathfrak{m}} &= (1\,\bar{5})(2\,\overline{10})(3\,\bar{2})(4\,\bar{8})(5\,\bar{4})(6\,\bar{1})(7\,\bar{9})(8\,\bar{6})(9\,\bar{7})(10\,\bar{3}).
\end{aligned}$$

As for the orientable maps, we can introduce the permutation

$$\varphi_{\mathfrak{m}} = \rho_{\mathfrak{m}} \mu_{\mathfrak{m}},$$

which describes the faces of $\mathfrak{m}$. Indeed, each face corresponds to two cycles: each cycle correspond to an exploration of the face in a different direction. For instance, in Figure 4.5, the same face is described by $(\bar{2}\,\bar{3}\,\overline{10})$ and $(3\,2\,10)$.

4.5.3 Maps on the orientation covering

We now explain how an orientable half-edge labelled map that is equipped with a involution without fixed point that reverses orientation (in a sense to be defined) can be seen as being a map on the orientation covering of some non-orientable surface. This will give the inverse of Construction 4.5.3.

Definition 4.5.6. *Let $I \subset \mathbf{N}^*$ be finite and $\hat{\mathfrak{m}}$ be an orientable half-edge labelled map with labels in $I \sqcup \bar{I}$. Let* inv *be a matching of $I \sqcup \bar{I}$. We say that* inv *is orientation-reversing if*

1. $$\varphi_{\mathfrak{m}} \circ \mathrm{inv} = \mathrm{inv} \circ \varphi_{\mathfrak{m}}^{-1} \text{ and } \alpha_{\mathfrak{m}} \circ \mathrm{inv} = \mathrm{inv} \circ \alpha_{\mathfrak{m}}^{-1},$$

2. *For each triple of cycles $(\pi, \pi', \pi'') \in \mathrm{Cycles}(\varphi_{\hat{\mathfrak{m}}}) \times \mathrm{Cycles}(\alpha_{\hat{\mathfrak{m}}}) \times \mathrm{Cycles}(\sigma_{\hat{\mathfrak{m}}})$, we have*

$$(\mathrm{inv} \circ \pi \circ \mathrm{inv})^{-1} \neq \pi, \quad (\mathrm{inv} \circ \pi' \circ \mathrm{inv})^{-1} \neq \pi', \text{ and } \quad (\mathrm{inv} \circ \pi'' \circ \mathrm{inv})^{-1} \neq \alpha_{\mathfrak{m}} \circ \pi'' \circ \alpha_{\mathfrak{m}}.$$

Remark 4.5.7. Note that this definition implies that

$$\begin{aligned}
(\mathrm{inv} \circ \varphi_{\mathfrak{m}} \circ \mathrm{inv})^{-1} &= \varphi_{\mathfrak{m}}\\
(\mathrm{inv} \circ \alpha_{\mathfrak{m}} \circ \mathrm{inv})^{-1} &= \alpha_{\mathfrak{m}}\\
(\mathrm{inv} \circ \sigma_{\mathfrak{m}} \circ \mathrm{inv})^{-1} &= \alpha_{\mathfrak{m}} \circ \sigma_{\mathfrak{m}} \circ \alpha_{\mathfrak{m}}.
\end{aligned}$$

In particular, condition 2 ensures that inv descends to an involution without fixed point on the sets of cycles of the three permutations $\sigma_{\mathfrak{m}}, \alpha_{\mathfrak{m}}$, and $\varphi_{\mathfrak{m}}$.

Using this notion, we can construct a bijection between flag-labelled non-orientable maps and maps on their orientation covering. It allows us to study non-orientable maps in terms of orientable maps.

Proposition 4.5.8. *Let S be a compact non-orientable surface, and $\hat{S}$ be its orientation covering. Fix a permutation φ. The construction below gives a bijection between the half-edge labelled maps $\hat{\mathfrak{m}}$ on $\hat{S}$ with face profile given by φ and equipped with an orientation-reversing matching* inv*, and the set of flag-labelled non-orientable maps $\mathfrak{m}$ on S with face profile φ and $\rho_{\mathfrak{m}} = \mathrm{inv}$.*

Furthermore, $\mathfrak{m}$ is described by the matchings defined below in (4.16), and any labelling on the vertices of $\hat{\mathfrak{m}}$ that is invariant by inv *descends to a labelling of $\mathfrak{m}$.*

For convenience, we assume that the set of labels is $[n, \bar{n}]$ for some $n \in \mathbf{N}^*$ and that $\varphi \in \mathfrak{S}([n, \bar{n}])$. Fix $\hat{\mathfrak{m}}$ with half-edge labelling function $\hat{\lambda}$. Let inv be an orientation reversing matching. We first explain how the involution inv induces an involution of the underlying surface. Consider any embedded graph $(\hat{\Gamma}, \hat{S}, \hat{\imath})$ representing $\hat{\mathfrak{m}}$. We define $\mathrm{inv}_{\hat{S}} \colon \hat{S} \to \hat{S}$ a continuous involution of the surface $\hat{S}$. We first define it on the vertices, then on the edges, and finally on the faces of the embedded graph $(\hat{\Gamma}, \hat{S}, \hat{\imath})$.

- Let u be a vertex of $\hat{\mathfrak{m}}$. It corresponds to a point $P \in \hat{S}$ and to a cycle π of $\sigma_{\hat{\mathfrak{m}}}$. By Definition 4.5.6, the permutation $(\alpha_{\hat{\mathfrak{m}}} \mathrm{inv} \pi \mathrm{inv} \alpha_{\hat{\mathfrak{m}}})^{-1}$ is a cycle of $\sigma_{\hat{\mathfrak{m}}}$. Hence, it corresponds to a vertex $\bar{u} \neq u$ of $\hat{\mathfrak{m}}$ and a point $P' \in \hat{S}$. We set $\mathrm{inv}_{\hat{S}}(P) = P'$. Similarly, $\mathrm{inv}_{\hat{S}}(P') = P$.

- Consider an edge of $\hat{\mathfrak{m}}$, made of two half-edges h_1 and h_2. It corresponds to a path e on $\hat{S}$. There is a unique edge made of the two half-edges $\bar{h}_1$ and $\bar{h}_2$ of labels $\mathrm{inv}(\hat{\lambda}(h_1))$ and $\mathrm{inv}(\hat{\lambda}(h_2))$ respectively. This second edge corresponds to a path e' on $\hat{S}$. We define $\mathrm{inv}_{\hat{S}}$ on e to be any homeomorphism from e to e' that makes $\mathrm{inv}_{\hat{S}}$ a continuous involution on $\hat{\imath}(\hat{\Gamma})$.

- Finally, consider a face f corresponding to a cycle π of $\varphi_{\hat{\mathfrak{m}}}$. There is a distinct face $\bar{f}$ corresponding to the cycle $(\mathrm{inv} \pi \mathrm{inv})^{-1}$ in $\varphi_{\hat{\mathfrak{m}}}$. We define $\mathrm{inv}_{\hat{S}}$ on f to be any homeomorphism from f to $\bar{f}$ that makes $\mathrm{inv}_{\hat{S}}$ a continuous involution. Such an extension exists: by the Jordan-Schönflies theorem (see for instance [MT01, Section 2.2]) we may construct a bijection extension of $\mathrm{inv}_{\hat{S}}$ on half of the faces, and define $\mathrm{inv}_{\hat{S}}$ on the other half of the faces so that it is an involution.

Lemma 4.5.9. *The map* $\mathrm{inv}_{\hat{S}}$ *is a continuous involution without fixed points that reverses the orientation.*

Proof. We constructed $\mathrm{inv}_{\hat{S}}$ to be an involution. Remark 4.5.7 implies that $\mathrm{inv}_{\hat{S}}$ has no fixed points.

To see that $\mathrm{inv}_{\hat{S}}$ is orientation-reversing, we consider any face f corresponding to a region D (homeomorphic to a disc) in $\hat{S}$ and a cycle π of $\varphi_{\mathfrak{m}}$. The half-edges around f are $h_1, \ldots, h_d$ in the clockwise orientation. The disk $\mathrm{inv}_{\hat{S}}(D) \subset \hat{S}$ corresponds to a face f' in $\hat{\mathfrak{m}}$ and to the cycle $\pi' = (\mathrm{inv} \pi \mathrm{inv})^{-1} \mathrm{inv} \neq \pi$. Let h'_i be the unique half-edge with label $\mathrm{inv}(\hat{\lambda}(h_i))$. We have

$$\begin{aligned}
&\pi(\hat{\lambda}(h_i)) = \hat{\lambda}(h_{i+1}) \\
&\pi'(\hat{\lambda}(h'_i)) = \mathrm{inv} \pi^{-1} \mathrm{inv}(\hat{\lambda}(h'_i)) = \mathrm{inv} \pi^{-1}(\hat{\lambda}(h_i)) = \mathrm{inv}(\hat{\lambda}(h_{i-1})) = \hat{\lambda}(h'_{i-1}).
\end{aligned}$$

Hence, $\mathrm{inv}_{\hat{S}}$ is orientation reversing in D. It is then orientation reversing globally. □

The quotient space $S = \hat{S}/\mathrm{inv}_{\hat{S}}$ is a surface, and the projection $p_{\hat{S}} \colon \hat{S} \to S$ is a double covering of S. The associated deck transformation is $\mathrm{inv}_{\hat{S}}$. By Lemma 4.5.9, $\mathrm{inv}_{\hat{S}}$ is an orientation-reversing involution. By Theorem 4.5.2, $\hat{S}$ is isomorphic to the orientation covering of S.

The projection p_S allows us to define a graph embedding in S. Denote by $V_{\hat{\Gamma}}$, $H_{\hat{\Gamma}}$, and $E_{\hat{\Gamma}}$ the sets of vertices, half-edges, and edges of $\hat{\Gamma}$. The new graph is Γ with sets of vertices, half-edges, and edges given by

$$\begin{aligned}
V_\Gamma &= \left\{ \{u, \bar{u}\} \colon u \in V_{\hat{\Gamma}} \right\}, \\
H_\Gamma &= \left\{ \{h, \bar{h}\} \colon h \in H_{\hat{\Gamma}} \right\}, \\
E_\Gamma &= \left\{ \{h_1, h_2\} \colon h_1 = \{g_1, \bar{g}_1\}, h_2 = \{g_2, \bar{g}_2\}, g_1, g_2 \in H_{\hat{\Gamma}} \right\}.
\end{aligned}$$

The graph embedding $\iota\colon \Gamma \to S$ is obtained by taking the image by $p_{\hat{S}}$ of $\hat{\iota}(\hat{\Gamma})$. Note that two vertices $\hat{\iota}(u), \hat{\iota}(\bar{u})$ in $\hat{S}$ have the same image by $p_{\hat{S}}$, and correspond to a unique vertex $\{u, \bar{u}\}$ in Γ. Similarly, each half-edge, edge, or face of the graph embedded in S have two preimages in $\hat{S}$.

Now, by definition, any two choices of $(\hat{\Gamma}, \hat{S}, \hat{\iota})$ and $(\tilde{\Gamma}, \tilde{S}, \tilde{\iota})$ in the class of $\hat{\mathfrak{m}}$ are isomorphic. If we denote by $\psi\colon \hat{S} \to \tilde{S}$ the orientation preserving homeomorphism between the two surfaces we have that $\mathrm{inv}_{\hat{S}}$ and $\psi^{-1}\mathrm{inv}_{\tilde{S}}\psi$ are homotopic (due to the possibly different choices to map corresponding edges and faces together). It implies that $\hat{S}/\mathrm{inv}_{\hat{S}}$ and $\tilde{S}/\mathrm{inv}_{\tilde{S}}$ are homeomorphic, and that $p_{\hat{S}}$ and $p_{\tilde{S}}$ are isomorphic coverings. This shows that we can define the map $\mathfrak{m}$ to be the isomorphism class of (Γ, S, ι). This construction is well-defined and does no depend on the choice of $(\hat{\Gamma}, \hat{S}, \hat{\iota})$. This concludes the construction: we have constructed from $\hat{\mathfrak{m}}$ a new map $\mathfrak{m}$, which is non-orientable (as its orientation covering is connected). We shall abuse notation and refer to $\hat{\mathfrak{m}}$ as the orientation covering of $\mathfrak{m}$.

Let us now describe the permutational model associated to $\mathfrak{m}$. We start by defining a flag-labelling function λ. Consider a flag $\mathfrak{f}$ in $\mathfrak{m}$, i.e. a side of a half-edge h (see Section 4.5.2). This flag has two preimages in the orientation covering $\hat{\mathfrak{m}}$, $\hat{\mathfrak{f}}_1$ and $\hat{\mathfrak{f}}_2$. In $\hat{\mathfrak{m}}$, only one of these two flags is on the left side of a half-edge h. This follows from the fact that inv reverses the orientation. We label the flag $\mathfrak{f}$ by $\hat{\lambda}(h)$, i.e. we set $\lambda(\mathfrak{f}) = \hat{\lambda}(h)$. The $2n$ flags are thus labelled by the elements of $[n, \bar{n}]$. We now define three matchings τ, μ, and ρ. We set

$$\tau_{\mathfrak{m}} = \alpha_{\hat{\mathfrak{m}}}\mathrm{inv}, \quad \rho_{\mathfrak{m}} = \mathrm{inv}, \text{ and } \mu_{\mathfrak{m}} = \tau\sigma_{\hat{\mathfrak{m}}}^{-1} = \mathrm{inv}\varphi_{\hat{\mathfrak{m}}}. \tag{4.16}$$

Definition 4.5.6 implies that these three permutations are indeed matchings: the fact that they are involutions follows from the first part of the definition, the fact that they do not have fixed point follow from the second part.

Lemma 4.5.10. *The triple of matchings (τ, ρ, μ) describes $\mathfrak{m}$.*

Proof. Let $\mathfrak{f}$ and $\mathfrak{f}'$ be two flags part of the same side of a same edge of $\mathfrak{m}$. These two flags are incident to a face f. Denote by h_1 (respectively h_1') the half-edge whose left side is a preimage of $\mathfrak{f}$ (resp. $\mathfrak{f}'$). Let $\mathfrak{f}_1$ and $\mathfrak{f}_1'$ the left sides of h_1 and h_1', and $\mathfrak{f}_2$ and $\mathfrak{f}_2'$ the right sides of the counterpart h_2 and h_2' of h_1 and h_1'. The continuous involution sends the pair of flags $\mathfrak{f}_1, \mathfrak{f}_2$ to $\mathfrak{f}_1', \mathfrak{f}_2'$. Hence, it sends the label of h to the label of h', i.e. $\mathrm{inv}(\lambda(\mathfrak{f})) = \lambda(\mathfrak{f}')$. On the other hand, we have by definition $\rho_{\mathfrak{m}}(\lambda(\mathfrak{f})) = \lambda(\mathfrak{f}')$. Thus, $\mathrm{inv} = \rho_{\mathfrak{m}}$.

Now, notice that $\hat{\lambda}(h_2') = \hat{\lambda}(\alpha_{\hat{\mathfrak{m}}}(h_2))$ is the label of the flag on the other side of $\mathfrak{f}$, i.e. $\tau_{\mathfrak{m}}(\lambda(\mathfrak{f})) = \alpha_{\hat{\mathfrak{m}}}\rho_{\mathfrak{m}}(\lambda(\mathfrak{f}))$. This means that

$$\tau_{\mathfrak{m}} = \alpha_{\hat{\mathfrak{m}}}\mathrm{inv} = \mathrm{inv}\alpha_{\hat{\mathfrak{m}}}.$$

Finally, let $\mathfrak{f}_3$ the other flag in the corner $\mathfrak{f}$ is part of. Its preimage at the left of a half-edge is in the same corner as h_2'. Hence we have

$$\lambda(\mathfrak{f}_3) = \mu(\lambda(\mathfrak{f})) = \sigma_{\hat{\mathfrak{m}}}\alpha_{\hat{\mathfrak{m}}}\mathrm{inv} = \sigma_{\hat{\mathfrak{m}}}\mathrm{inv}\alpha_{\hat{\mathfrak{m}}} = \mathrm{inv}\alpha_{\mathfrak{m}}\sigma_{\hat{\mathfrak{m}}}^{-1} = \mathrm{inv}\varphi_{\hat{\mathfrak{m}}}.$$

□

The faces of $\mathfrak{m}$ are then described by the permutation $\varphi_{\mathfrak{m}} = \rho\mu$. We have

$$\varphi_{\mathfrak{m}} = \mathrm{inv}\mathrm{inv}\varphi_{\hat{\mathfrak{m}}} = \varphi_{\hat{\mathfrak{m}}}.$$

Proof of Proposition 4.5.8. We explained how to construct $\mathfrak{m}$ from $\hat{\mathfrak{m}}$, let us now give the inverse construction.

Let $\mathfrak{m}$ be a non-orientable map on S which is flag-labelled by $\lambda\colon \mathrm{Fl}_{\mathfrak{m}} \to [n]$. We explained in Construction 4.5.3 how to construct a map $\hat{\mathfrak{m}}$ on the orientation covering $\hat{S}$ of S. The orientation covering $\hat{S}$ is endowed with an orientation-reversing involution $\mathrm{inv}_{\hat{S}}$. The half-edges of $\hat{\mathfrak{m}}$ are naturally labelled: the flag $\hat{\mathfrak{f}}$ at the left side of a half-edge h has one image by p, $\mathfrak{f}$. We set $\hat{\lambda}(h) = \lambda(\mathfrak{f})$. The continuous involution $\mathrm{inv}_{\hat{S}}$ induces an involution inv on the labels of the half-edges of $\hat{\mathfrak{m}}$ as follows. Let

h be a half-edge in $\hat{\mathfrak{m}}$ and h' its counterpart. The half-edge h is incident to a face f and the image by $\mathrm{inv}_{\hat{S}}$ of h', h'' is incident to $\mathrm{inv}_{\hat{S}}(f)$. We set $\mathrm{inv}(\hat{\lambda}(h)) = \hat{\lambda}(h'')$. This coincides with $\rho_{\mathfrak{m}}$. We now explain why this involution must be an orientation-reversing matching. Indeed, if condition 1 of Definition 4.5.6 were not satisfied, $\mathrm{inv}_{\hat{S}}$ would not be orientation-reversing. If condition 2 were not satisfied, there would be a face, edge, or vertex whose image by $\mathrm{inv}_{\hat{S}}$ would be itself. By Brouwer fixed point Theorem, $\mathrm{inv}_{\hat{S}}$ would have a fixed point: it contradicts the fact that $\mathrm{inv}_{\hat{S}}$ is a continuous involution without fixed point.

If a labelling ℓ of the vertices of $\hat{\mathfrak{m}}$ is invariant by inv, then for any vertex v in $\mathfrak{m}$, its two preimages in $\hat{\mathfrak{m}}$ have the same label and we can label v in a well-defined way. □

4.5.4 Cutting and gluing suitably labelled maps

The goal of this Section is to define a mapping from the set of suitably labelled maps with two local minima to the set of maps on $\mathbb{RP}^2$. The procedure starts by choosing a path in a suitably labelled map. We start by describing the procedure with a quite general choice of path. We obtain an injective mapping between sets of suitably labelled maps equipped with a curve.

We give some definition regarding what we mean by a path in a map.

Definition 4.5.11. *A **path** of length $l \geq 1$ is a sequence of half-edges $\boldsymbol{g} = (g_1, \dots, g_{2l})$, with g_{2i-1}, g_{2i} the two half-edges of a same edge for all $i = 1, \dots, l$, and g_{2i}, g_{2i+1} incident to the same vertex for $i = 1, \dots, l-1$. The length of the path is $\#\boldsymbol{g} = l$. The inverse of $\boldsymbol{g}$ is the path $\boldsymbol{g^{-1}}$:*

$$\boldsymbol{g^{-1}} = (g_{2l}, \dots, g_1).$$

*We say a path is a **loop** if $\mathrm{vert}(g_1) = \mathrm{vert}(g_{2l})$. A path is **simple** if $\mathrm{vert}(g_{2i}) \neq \mathrm{vert}(g_{2j})$ and $\mathrm{vert}(g_{2i-1}) \neq \mathrm{vert}(g_{2j-1})$ for all $i \neq j$.*

The concatenation of two paths $\boldsymbol{g}$ and $\boldsymbol{h}$ such that $\mathrm{vert}(g_{2\#\boldsymbol{g}}) = \mathrm{vert}(h_1)$ is

$$\boldsymbol{g} \sqcup \boldsymbol{h} = (g_1, \dots, g_{2\#\boldsymbol{g}}, h_1, \dots, h_{2\#\boldsymbol{h}}).$$

Finally, if $\boldsymbol{g}$ is simple, and if u and v are two vertices such that $u = \mathrm{vert}(g_{2p+1})$ and $v = \mathrm{vert}(g_{2q})$, we denote the subpath of $\boldsymbol{g}$ from u to v by

$$\boldsymbol{g}|_{u \to v} = (g_{2p+1}, \dots, g_{2q}).$$

An important assumption on some of the paths we consider is that they are *good* paths.

Definition 4.5.12. *Let $\boldsymbol{g} = (g_i)_{1 \leq i \leq 2l}$ be a simple path of length l in a suitably labelled map $(\mathfrak{m}, \ell)$. Set $v_0 = |g_1$ and $v_i = \mathrm{vert}(g_{2i})$ for $i \in [l]$. We say $\boldsymbol{g}$ is a **good path** if $v_0 \neq v_l$ and the function*

$$\ell_{\boldsymbol{g}} : \begin{cases} \{0, 1, \dots, l\} & \to \mathbf{N} \\ i & \mapsto \ell(v_i) \end{cases}$$

has exactly two local minima, achieved at $i = 0$ and $i = l$, with $\ell_{\boldsymbol{g}}(0) = \ell_{\boldsymbol{g}}(l)$, and either one local maximum, or two local maxima attained at consecutive values.

*We say a simple loop $\boldsymbol{g}$ is a **good loop** if it can be written as the concatenation $\boldsymbol{g} = \boldsymbol{g}_1 \sqcup \boldsymbol{g}_2$ of two good paths $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ with $\ell_{\boldsymbol{g_1}} = \ell_{\boldsymbol{g_2}}$.*

Example 4.5.13. A simple path $\boldsymbol{g}$ with $\ell_{\boldsymbol{g}}$ given by

$$(\ell_{\boldsymbol{g}}(i))_{i=0,1,\dots,l} = (0, 1, 2, 2, 1, 0)$$

is good. However, if

$$(\ell_{\boldsymbol{g}}(i))_{i=0,1,\dots,l} = (0, 1, 2, 3, 2, 3, 2, 1, 0),$$

it is not good.

We now describe several transformations that can be applied to a suitably labelled map $(\mathfrak{m}, \ell)$ with a distinguished good path $\boldsymbol{g}$.

Opening a slit

The first transformation corresponds to adding a new face to $\mathfrak{m}$. This new face will be seen as a boundary. In the process, we will add new faces, with new labels that will be barred integer, for convenience. The new half-edges we add will also be denoted with a bar. We assume that if a half-edge h is labelled by i, then $\bar{h}$ is labelled by $\bar{i}$.

Definition 4.5.14. *A face is simple if when going around it, each edge is encountered exactly once. A* ***map with boundary*** $(\mathfrak{m}, f)$ *is a map* $\mathfrak{m}$ *with a distinguished simple face* f*. A boundary of* $\mathfrak{m}$ *is any choice of simple loop* $\boldsymbol{g} = (g_1, \ldots, g_{2l})$ *such that* f *is incident to* g_{2j-1} *for* $j \in [l]$*.*

If $\mathfrak{m}$ *is half-edge labelled with labels in a set* I*, we denote by* $\varphi_{\mathfrak{m}}(f)$ *the cycle representing* f *and we set*

$$\varphi_{\mathfrak{m}\backslash f} = \varphi_{\mathfrak{m}} \left(\varphi_{\mathfrak{m}}(f)\right)^{-1}|_{I \backslash \operatorname{Supp} \varphi_{\mathfrak{m}}(f)}.$$

Maps with boundaries can be seen as being embedded in other maps.

Definition 4.5.15. *Let* $(\mathfrak{m}, f)$ *be a half-edge labelled map with boundary and* $\hat{\mathfrak{m}}$ *be a half-edge labelled map. We say that* $\mathfrak{m}$ *is embedded in* $\hat{\mathfrak{m}}$ *if*

$$\operatorname{Cycles}(\varphi_{\mathfrak{m}\backslash f}) \subset \operatorname{Cycles}(\varphi_{\hat{\mathfrak{m}}}) \text{ and } \operatorname{Cycles}(\alpha_{\mathfrak{m}}) \subset \operatorname{Cycles}(\alpha_{\hat{\mathfrak{m}}}).$$

We now describe the construction. An example is depicted in Figures 4.6 and 4.7.

Construction 4.5.16. *Consider a suitably labelled map* $(\mathfrak{m}, \boldsymbol{g})$ *and* $\boldsymbol{g}$*, a good path of length* $l \geq 1$*. Let* $v_0 = \operatorname{vert}(g_1)$ *and* $v_i = \operatorname{vert}(g_{2i})$ *for* $i \in [l]$*. Each* $\pi \in \operatorname{Cycles}(\sigma_{\mathfrak{m}})$ *corresponds to a vertex of* $\mathfrak{m}$*. For each such cycle, we proceed as follows.*

- *If* π *corresponds to none of the* $v_i, i = 0, 1, \ldots, l$*, we leave it unchanged.*
- *If* π *corresponds to a vertex* v_i *for* $i \in [l-1]$*, it can be written* $(g_{2i}\, u_1\, \ldots\, u_d\, g_{2i+1}\, v_1\, \ldots\, v_{d'})$*. We replace* π *by*
$$\pi' = (u_1\, \ldots\, u_d\, g_{2i+1}\, \overline{g_{2l-2i+2}})(v_1\, \ldots\, v_d\, g_{2i}\, \overline{g_{2l-2i-1}}).$$
- *If* π *corresponds to* v_0*, it can be written* $(g_1\, u_1\, \ldots\, u_d)$*. We replace* π *by*
$$\pi' = (g_1\, u_1\, \ldots\, u_d\, \overline{g_{2l-1}}).$$
- *If* π *corresponds to* v_l*, it can be written* $(g_{2l}\, v_1\, \ldots\, v_d)$*. We replace* π *by*
$$\pi' = (\overline{g_2}\, v_1\, \ldots\, v_d\, g_{2l}).$$

We obtain a new permutation σ'*. We set*

$$\varphi' = \varphi_{\mathfrak{m}} \tilde{\varphi}, \text{ with } \tilde{\varphi} = (\overline{g_{2l-1}}\, \overline{g_{2l-3}}\, \ldots\, \overline{g_1}\, \overline{g_2}\, \overline{g_4}\, \ldots\, \overline{g_{2l}}),$$

and $\alpha' = (\varphi')^{-1}(\sigma')^{-1}$*. The permutations* (σ', α') *determine a half-edge labelled map* $\mathfrak{m}'$ *with a marked face – represented by the cycle* $\tilde{\varphi}$*. The vertex-labelling* ℓ *of* $\mathfrak{m}$ *induces a vertex-labelling* ℓ' *of* $\mathfrak{m}'$*: each vertex* v' *of* $\mathfrak{m}'$ *is constructed from a vertex* v *of* $\mathfrak{m}$*, we set* $\ell'(v') := \ell(v)$*.*

Note that α' is a matching by construction: the edges in $\boldsymbol{g}$, which are represented by a cycle $(g_{2i-1}\, g_{2i})$ of $\alpha_{\mathfrak{m}}$, become the pair of edges represented by $(g_{2i-1}\, \overline{g_{2l-2i+2}})(g_{2i}\, \overline{g_{2l-2i+1}})$ for $i \in [l-1]$.

Lemma 4.5.17. *The map* $(\mathfrak{m}', \ell')$ *constructed in Construction 4.5.16 is a suitably labelled map with boundary* f*. The boundary of* f *is a good loop.*

Proof. We start by showing that the labelling is suitable. Let v and v' two vertices of $\mathfrak{m}'$ which are connected by an edge. These two vertices are constructed from vertices $\hat{v}$ and $\hat{v}'$ in $\mathfrak{m}$. By construction, if v and v' are connected by an edge, so are $\hat{v}$ and $\hat{v}'$. The fact that $(\mathfrak{m}, \ell)$ is a suitably labelled map thus implies that $(\mathfrak{m}', \ell')$ is a suitably labelled map.

In Construction 4.5.16, each edge of $\boldsymbol{g}$ gets duplicated. We can choose a boundary of f to be a loop $\boldsymbol{g}' = (g'_i)_{i\in[4l]}$ with $g'_i = g_i$ and $g'_{2l+i} = g_{2l-i}$ for $i \in [2l]$. As $\boldsymbol{g}$ is a good loop, we have $\ell_{\boldsymbol{g}}(i) = \ell_{\boldsymbol{g}}(l-i)$ for $i = 0, 1, \ldots, l$. It follows that $\boldsymbol{g}'$ is a good loop. $\square$

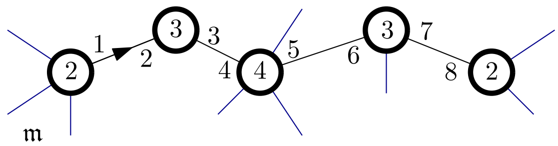


Figure 4.6: Example of a good path.

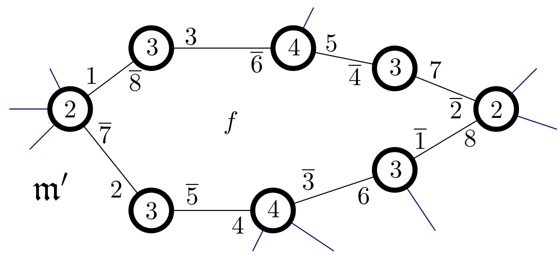


Figure 4.7: Opening of a new face along the good path.

The mirror map

Given a half-edge labelled, suitably labelled map $(\mathfrak{m}, \ell)$, we may construct the "mirror map", obtained after changing the orientation of all the vertices in $\mathfrak{m}$.

This reversing of the orientation is encoded at the level of the permutation by the following transformation.

Definition 4.5.18. *Let $n \in \mathbf{N}^*$, $I \subset [n, \bar{n}]$, and $\sigma \in \mathfrak{S}(I)$. Each cycle $\pi \in \mathrm{Cycles}(\sigma)$ can be written*

$$\pi = (u_1 \ \ldots \ u_d).$$

We set

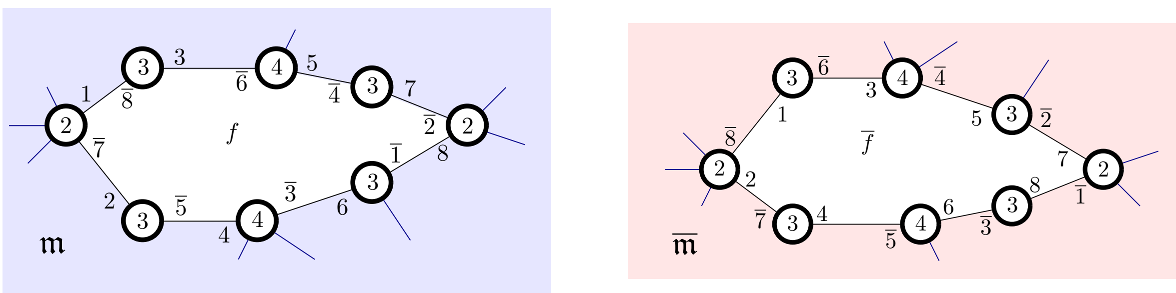
$$\bar{\pi} = (\overline{u_d} \ \ldots \ \overline{u_1}),$$

and

$$\bar{\sigma} = \prod_{\pi \in \mathrm{Cycles}(\sigma)} \bar{\pi} \in \mathfrak{S}(\bar{I}).$$

We have in particular that for two permutations σ_1 and σ_2,

$$\bar{\sigma}_1 \bar{\sigma}_2 = \overline{\sigma_2 \sigma_1} \text{ and } \bar{\sigma}_1^{-1} = \overline{\sigma_1^{-1}}.$$



(a) (b)

Figure 4.8: (a) The map of Figure 4.7, (b) its mirror map.

The map $\bar{\mathfrak{m}}$ determined by $(\bar{\alpha}_{\mathfrak{m}}^{-1} \bar{\varphi}_{\mathfrak{m}}^{-1}, \bar{\alpha}_{\mathfrak{m}})$ is called the mirror of $\mathfrak{m}$. We denote by $\mathrm{inv}_{\mathfrak{m}}$ the mapping from the set of half-edges of $\mathfrak{m}$ and the set of half-edges of $\bar{\mathfrak{m}}$ sending a half-edge to its mirror image. In particular, if a half-edge h is labelled by u, the half-edge $\mathrm{inv}_{\mathfrak{m}}(h)$ is labelled by $\bar{u}$. We abuse notation and denote by $\mathrm{inv}_{\mathfrak{m}}(v)$ and $\mathrm{inv}_{\mathfrak{m}}(f)$ the mirror image in $\mathfrak{m}'$ of a vertex v of a face f in $\mathfrak{m}$. Finally, we define $\bar{\ell}$ by

$$\bar{\ell}(\mathrm{inv}_{\mathfrak{m}}(v)) = \ell(v) \text{ for any vertex } v \text{ of } \mathfrak{m}.$$

Lemma 4.5.19. *The constructed map $(\bar{\mathfrak{m}}, \bar{\ell})$ is a suitably labelled map.*

Proof. The mirror construction descends to a bijection between the underlying graphs of $\mathfrak{m}$ and $\bar{\mathfrak{m}}$ that preserves the labelling of the vertices. The fact that $(\mathfrak{m}, \ell)$ is a suitably labelled map implies the result. □

Note that the vertex permutation of $\mathfrak{m}'$ is

$$\sigma_{\bar{\mathfrak{m}}} = \alpha_{\bar{\mathfrak{m}}}^{-1}\varphi_{\bar{\mathfrak{m}}}^{-1} = \bar{\alpha}_{\mathfrak{m}}^{-1}\bar{\varphi}_{\mathfrak{m}}^{-1} = (\bar{\varphi}_{\mathfrak{m}}\bar{\alpha}_{\mathfrak{m}})^{-1} = (\overline{\alpha_{\mathfrak{m}}\varphi_{\mathfrak{m}}})^{-1} = \bar{\alpha}_{\mathfrak{m}}\left(\overline{\varphi_{\mathfrak{m}}\alpha_{\mathfrak{m}}}\right)^{-1}\bar{\alpha}_{\mathfrak{m}}^{-1} = \bar{\alpha}_{\mathfrak{m}}\bar{\sigma}_{\mathfrak{m}}\bar{\alpha}_{\mathfrak{m}}^{-1}. \quad (4.17)$$

Gluing along a face

The last construction we define is how to glue a map with boundary to its mirror map, along their distinguished faces. We use the two previous constructions. Fix a suitably labelled map $(\mathfrak{m}_0, \ell_0)$ and a good path $\boldsymbol{g_0}$ in $\mathfrak{m}_0$. We use Construction 4.5.16 to obtain a suitably labelled map $(\mathfrak{m}, \ell)$ with boundary face f. Let $(\bar{\mathfrak{m}}, \bar{\ell})$ be the mirror map of $(\mathfrak{m}, \ell)$. It is a map with boundary face $\mathrm{inv}_{\mathfrak{m}}(f)$. Let $\boldsymbol{g}$ be a choice of boundary of f in $\mathfrak{m}$ such that $\boldsymbol{g} = (g_i)_{i\in[4l]}$ is a good loop. There is a canonical way to glue $\mathfrak{m}$ and $\bar{\mathfrak{m}}$ along f. The labels of the half-edges of the boundary of $\mathfrak{m}$ and $\bar{\mathfrak{m}}$ were constructed to be the same. There is a natural way to identify an edge on the boundary of $\mathfrak{m}$ to an edge on the boundary of $\bar{\mathfrak{m}}$.

We define the boundary permutation

$$\alpha_{\partial f} = \prod_{i=1}^{l}(g_{2i-1}\,\overline{g_{2l-2i+2}})(g_{2i}\,\overline{g_{2l-2i+1}}).$$

We then notice that the labels of the half-edges that are not on the boundary are integers in $\mathfrak{m}$ and barred integers in $\bar{\mathfrak{m}}$. This means that $\varphi_{\mathfrak{m}}$ and $\varphi_{\bar{\mathfrak{m}}}$ have disjoint support, and that two cycles in $\mathrm{Cycles}(\alpha_{\mathfrak{m}}) \setminus \mathrm{Cycles}(\partial_{\mathfrak{m}})$ and $\mathrm{Cycles}(\alpha_{\bar{\mathfrak{m}}}) \setminus \mathrm{Cycles}(\partial_{\mathfrak{m}})$ have disjoint support as well. Finally, we have

$$\mathrm{Cycles}(\alpha_{\mathfrak{m}}) \cap \mathrm{Cycles}(\alpha_{\bar{\mathfrak{m}}}) = \mathrm{Cycles}(\alpha_{\partial f}).$$

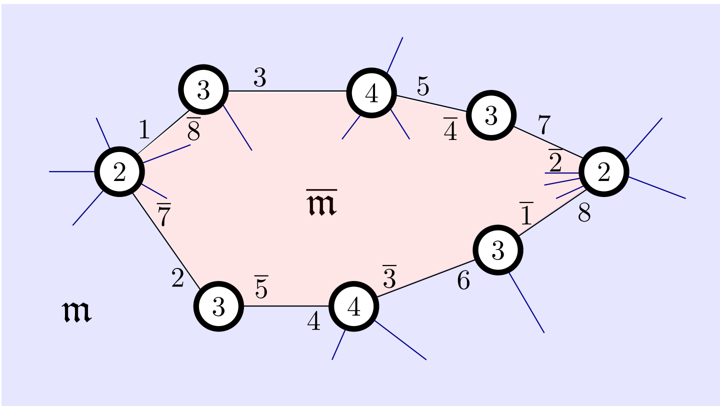


Figure 4.9: Map obtained by gluing a map to its mirror map.

We thus define the permutations

$$\begin{aligned}\varphi_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}} &= \varphi_{\mathfrak{m}}\varphi_{\bar{\mathfrak{m}}}\\ \alpha_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}} &= \alpha_{\mathfrak{m}}\alpha_{\bar{\mathfrak{m}}}\alpha_{\partial f}\\ \sigma_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}} &= \alpha_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}^{-1}\varphi_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}^{-1},\end{aligned}$$

The pair of permutations $(\sigma_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}, \alpha_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}})$ defines a map. By construction, the maps with boundary $(\mathfrak{m}, f)$ and $(\bar{\mathfrak{m}}, \mathrm{inv}_{\mathfrak{m}}(f))$ are embedded in $\mathfrak{m} \sqcup \bar{\mathfrak{m}}$. Furthermore, every face (and hence vertex) of $\mathfrak{m} \sqcup \bar{\mathfrak{m}}$ can be seen as being part of either $\mathfrak{m}$ or $\bar{\mathfrak{m}}$ (or both, in the case of vertices). Thus, for every vertex v of $\mathfrak{m} \sqcup \bar{\mathfrak{m}}$ we set

$$\ell_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}(v) = \begin{cases}\ell(v) & \text{if } v \text{ is part of } \mathfrak{m}\\ \bar{\ell}(v) & \text{if } v \text{ is part of } \bar{\mathfrak{m}}.\end{cases}$$

Any boundaries of $\mathfrak{m}$ and $\bar{\mathfrak{m}}$ are made of the same half-edges (possibly in a different cyclic order). They form a loop $\boldsymbol{g}_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}$. It can be chosen to be a good loop since the boundary of $\mathfrak{m}$ is a good loop by Lemma 4.5.17.

Lemma 4.5.20. *The map* $(\mathfrak{m} \sqcup \bar{\mathfrak{m}}, \ell_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}})$ *is a suitably labelled map. It is orientable and its genus is twice the genus of* $\mathfrak{m}$.

Proof. Each edge of $\mathfrak{m} \sqcup \bar{\mathfrak{m}}$ can be seen as being part of either $\mathfrak{m}$ or $\bar{\mathfrak{m}}$ (or both). Hence, if an edge e between vertices v and v' can be seen as being part of, say, $\mathfrak{m}$, we have

$$\left|\ell_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}(v) - \ell_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}(v')\right| = \left|\ell(v) - \ell(v')\right| \leq 1.$$

Furthermore, the minimum label of a vertex in $\mathfrak{m} \sqcup \bar{\mathfrak{m}}$ is 0.

The vertices and edges that are not part of this loop are part of exactly one of the embedded maps $\mathfrak{m}$ and $\bar{\mathfrak{m}}$. The vertices and edges part of this loop are part of both $\mathfrak{m}$ and $\bar{\mathfrak{m}}$. As it is a simple loop, the number of vertices and edges that are part of $\boldsymbol{g}_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}$. Considering that $\mathfrak{m}$ and $\bar{\mathfrak{m}}$ have the same genus, Euler formula implies that the genus of $\mathfrak{m} \sqcup \bar{\mathfrak{m}}$ is twice the genus of $\mathfrak{m}$. ☐

The mirror map $\mathrm{inv}_{\mathfrak{m}}$ allows us to define an involution inv on the set of labels of the half-edges of $\mathfrak{m} \sqcup \bar{\mathfrak{m}}$. It is defined as follows. Let h be a half-edge in $\mathfrak{m} \sqcup \bar{\mathfrak{m}}$ with label $i = \lambda_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}(h)$. We set

$$\mathrm{inv}(i) = \begin{cases} \lambda_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}(\mathrm{inv}_{\mathfrak{m}}(h)) & \text{if } h \text{ is in } \mathfrak{m} \\ \lambda_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}(\mathrm{inv}_{\mathfrak{m}}^{-1}(h)) & \text{if } h \text{ is in } \bar{\mathfrak{m}} \end{cases} = \bar{i}.$$

Lemma 4.5.21. *The mapping* inv *is an orientation-reversing matching (in the sense of Definition 4.5.6).*

Proof. By definition of $\bar{\alpha}_{\mathfrak{m}}$ (recall Definition 4.5.18), we have

$$\mathrm{inv}\varphi_{\mathfrak{m}}\mathrm{inv} = \varphi_{\bar{\mathfrak{m}}}^{-1} \quad \text{and} \quad \mathrm{inv}\alpha_{\mathfrak{m}}\mathrm{inv} = \bar{\alpha}_{\mathfrak{m}}^{-1} = \alpha_{\bar{\mathfrak{m}}}.$$

We also have $\mathrm{inv}\alpha_{\partial f}\mathrm{inv} = \alpha_{\partial f}$. This gives

$$\mathrm{inv}\varphi_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}} = \varphi_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}^{-1}\mathrm{inv} \quad \text{and} \quad \mathrm{inv}\alpha_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}^{-1} = \alpha_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}\mathrm{inv}.$$

Using this, we have

$$\mathrm{inv}\sigma_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}^{-1}\mathrm{inv} = \mathrm{inv}\varphi_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}\alpha_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}\mathrm{inv} = \varphi_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}^{-1}\alpha_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}} = \alpha_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}\sigma_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}\alpha_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}.$$

By construction, the involution sends the label of a half-edge h to the label of another half-edge h' which is neither incident to the same vertex, nor incident to the same face, nor part of the same edge. It is easy to see for the faces and the edges: the conjugation by inv replace all the elements of a cycle by their barred versions. The cycle of the faces have support in either the integers or the barred integers, it is also the case for the edges that are not on $\boldsymbol{g}_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}$. We can then check that no cycle $\pi \in \alpha_{\partial f}$ satisfies $\mathrm{inv}\pi\mathrm{inv} = \pi$. For the vertices, assume that there exists a cycle $\pi \in \mathrm{Cycles}(\sigma_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}})$ such that $(\alpha_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}\mathrm{inv}\sigma_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}\alpha_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}})^{-1} = \pi$. Necessarily, the vertex corresponding to this cycle is on $\boldsymbol{g}_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}$. If it is not the case the support of π is in the integers or the barred integers, without the elements of the support of $\alpha_{\partial f}$. We can then proceed as for the faces. Let i be the only element in $\mathrm{Supp}\,\pi \cap \mathrm{Supp}\,\alpha_{\partial f} \cap \mathbf{N}^*$. We have that $\pi^{-1}(i) \neq i$ (π is incident to at least two edges) and thus $\mathrm{inv}\alpha_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}(i) = \mathrm{inv}\alpha_{\partial f}(i) \in \mathrm{Supp}\,\pi \cap \mathrm{Supp}\,\alpha_{\partial f} \cap \mathbf{N}^*$. However, this set is the singleton $\{i\}$ and

$$\mathrm{inv}\alpha_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}(i) = i.$$

The permutation $\mathrm{inv}\alpha_{\mathfrak{m}\sqcup\bar{\mathfrak{m}}}$ has no fixed point, we have reached a contradiction. ☐

Let us sum up what has been achieved so far. Starting from a suitably labelled map $(\tilde{\mathfrak{m}}, \ell)$ an a good path $\tilde{\boldsymbol{g}}$, we produced a suitably labelled map with boundary $(\hat{\mathfrak{m}}_+, \ell_+)$. We glue this map to its mirror image $\hat{\mathfrak{m}}_-$ to obtain a new suitably labelled map $(\hat{\mathfrak{m}}, \hat{\ell})$ in which both $\hat{\mathfrak{m}}_+$ and $\hat{\mathfrak{m}}_-$ are embedded. The map $(\hat{\mathfrak{m}}, \hat{\ell})$ is equipped with a good loop $\hat{\boldsymbol{g}}$. Finally, thanks to Lemma 4.5.21, we can use the construction of Section 4.5.3 to see $\hat{\mathfrak{m}}$ as being a map on the orientation covering of a non-orientable map $\mathfrak{m}$.

Proposition 4.5.22. *The mapping just described, which associates, to a suitably labelled map $(\tilde{\mathfrak{m}}, \tilde{\ell})$ equipped with a good path $\tilde{\boldsymbol{g}}$, the suitably labelled map $(\hat{\mathfrak{m}}, \hat{\ell})$, is injective.*

Proof. This follows from the fact that there is a right inverse to this mapping which we now describe. From $(\hat{\mathfrak{m}}, \hat{\ell})$ we can recover the embedded map $(\hat{\mathfrak{m}}_+, \hat{\ell}_+)$: it is the unique embedded map whose non-boundary faces are exactly the ones with labels in the integers. Then, we glue the edges along the boundary face together. We glue together the two edges incident to each vertex of minimal label on the boundary. We do this repeatedly until there is no boundary face. The labels erased during this procedure are barred integers that where added in the first step (when opening the slit) ☐

The construction above depended on a choice of good path $\tilde{\boldsymbol{g}}$ in $\tilde{\mathfrak{m}}$. We now explain how to choose it in a canonical way.

4.5.5 Choosing a path in a map

We now explain a way to choose a path $\tilde{\boldsymbol{g}}$ in a suitably labelled map $(\tilde{\mathfrak{m}}, \tilde{\ell})$, and characterize the image of this path in the glued map $(\hat{\mathfrak{m}}, \hat{\ell})$. We will show that when considering a map on an orientation covering, there is a canonical choice of loop, which we call equilibrium loop. Starting from this Section, we assume that $\tilde{\mathfrak{m}}$ is planar (and thus, so is $\hat{\mathfrak{m}}$).

Local roots and leftmost paths

Definition 4.5.23. *Let $\mathfrak{m}$ be a suitably labelled map with labelled half-edges. The half-edge h^* with minimal label among those attached to a vertex of label 0 is said to be the **root**. The vertex v^* it is attached to is the **root vertex**.*

*A **local root** at a vertex v is the choice of a half-edge incident to v.*

The notion of a local root allows us to define an ordering of the half-edges at a vertex.

Definition 4.5.24. *Let v be a vertex with a local root h. Let $h = h_1, h_2, \dots, h_d$ be the half-edges around v in the clockwise order. We say that h_i is to the left of h_j if $i < j$.*

This ordering of the half-edges defines an ordering of the paths starting at a vertex equipped with a local root.

Definition 4.5.25. *Consider two paths $\boldsymbol{g} = (g_i)_{1\le i\le 2l}$ and $\boldsymbol{g'} = (g'_i)_{1\le i\le 2l'}$ with same starting vertex $v = \mathrm{vert}(g_1) = \mathrm{vert}(g'_1)$. Assume that v is equipped with a local root h. By convention, set $g_0 = g'_0 = h$. If there exists i, the first index such that $g_{2i+1} \neq g'_{2i+1}$ then taking $g_{2i} = g'_{2i}$ to be the local root at $v' = \mathrm{vert}(g_{2i}) = \mathrm{vert}(g'_{2i})$, we say that $\boldsymbol{g}$ is at the left of $\boldsymbol{g'}$ if g_{2i+1} is to the left of g'_{2i+1}. If there are no such i, then the shortest of the two paths is said to be to the left of the other.*

Note that this ordering of the paths defines a total order of the paths started at a locally rooted map.

Definition 4.5.26. *A geodesic between two vertices v and v' is a path of shortest length (for the graph distance) between v and v'.*

Construction 4.5.27. *Consider a map $\mathfrak{m}$ with labelled half-edges. Let us explain how the root h^* at the root vertex v^* induces a choice of local root for each vertex v in $\mathfrak{m}$. Among the geodesics from $v*$ to v, there is a leftmost geodesic $\boldsymbol{g} = (g_i)_{1\le i\le 2d}$. We choose the local root at v to be the half-edge following g_{2d} in the clockwise orientation around v.*

Furthermore, the (global) root allows to order the vertices.

Definition 4.5.28. *Let v^* be the root vertex, and v, v' two distinct vertices. Let $\boldsymbol{g}$ and $\boldsymbol{g'}$ be respectively the leftmost geodesic from v^* to v and from v^* to v'. We say that v is to the left of v' if $\boldsymbol{g}$ is to the left of $\boldsymbol{g'}$.*

Leftmost good geodesics

Consider a planar map $(\tilde{\mathfrak{m}}, \tilde{\ell}) \in \mathcal{S}_n$ with two local minima, v^* and v°. Assume that v^* is the root vertex, and thus $0 = \tilde{\ell}(v^*) \leq \tilde{\ell}(v^\circ)$. We now distinguish a path $\tilde{\boldsymbol{g}}$ in $\tilde{\mathfrak{m}}$. To do so we introduce another notion.

Definition 4.5.29. *A **good geodesic** starting from a vertex v is a path from v to a distinct vertex v' with $\tilde{\ell}(v) = \tilde{\ell}(v')$, of minimal length among the paths from v to a distinct vertex with label $\tilde{\ell}(v)$.*

Note that a good geodesic in a map with two local minima is in particular a good path.

We define $\tilde{\boldsymbol{h}}$ to be the leftmost geodesic from v^* to v°, with respect to the ordering given by the global root at v^*. As v° is a local minimum, there is a unique vertex $v^\bullet \neq v^\circ$ encountered by $\tilde{\boldsymbol{h}}$ such that $\tilde{\ell}(v^\circ) = \tilde{\ell}(v^\bullet)$. We may have $v^\bullet = v^*$ if $\tilde{\ell}(v^\circ) = 0$. We denote by $\tilde{\boldsymbol{g}}$ the subpath of $\tilde{\boldsymbol{h}}$ from $v^\bullet$ to v°.

Lemma 4.5.30. *The path $\tilde{\boldsymbol{g}}$ – up to reorienting it from v° to $v^\bullet$ – is a good geodesic from v°.*

Proof. Assume that there is a vertex u' with label $\ell(u') = \ell(v^\circ)$ and a path $\boldsymbol{h'}$ between v° and u' such that the length of $\boldsymbol{h'}$ is strictly smaller than $\tilde{\boldsymbol{g}}$. As u' is either v^* or not a local minimum, there is a path of length $\tilde{\ell}(u') = \tilde{\ell}(v^\circ)$ between v^* and u'. We can thus construct a path strictly shorter than $\tilde{\boldsymbol{h}}$ between v^* and v°. This contradicts the fact that $\tilde{\boldsymbol{h}}$ is a geodesic. ☐

Equilibrium loops

Fix a suitably labelled map $(\hat{\mathfrak{m}}, \hat{\ell})$, equipped with an orientation-reversing matching inv that preserves the labels, i.e. $\hat{\ell} \circ \mathrm{inv} = \hat{\ell}$. In this section, we explain how to choose in a unique way a good loop in $\hat{\mathfrak{m}}$. The construction is depicted on Figure 4.10. We denote by v^* the root of $\hat{\mathfrak{m}}$ and by $\bar{v}^* = \mathrm{inv}(v^*)$ its image by the involution.

Construction 4.5.31. *Let $\hat{\boldsymbol{h}}$ be the leftmost geodesic from v^* to $\bar{v}^*$, and $\mathrm{inv}(\hat{\boldsymbol{h}})$ be the path obtained from $\hat{\boldsymbol{h}}$ by applying inv to each of its half-edges. Let k be the number of vertices in $\hat{\boldsymbol{h}}$ that are also in $\mathrm{inv}(\hat{\boldsymbol{h}})$. Since $\mathrm{inv}(v^*) = \bar{v}^*$, we have $k \geq 2$. Let $u_1 = v^*, u_2, \ldots, u_k = \bar{v}^*$ be these vertices, in the order they are encountered by $\hat{\boldsymbol{h}}$. Note that for all $i \in [k]$, $\mathrm{inv}(u_i)$ is also both in $\hat{\boldsymbol{h}}$ and $\mathrm{inv}(\hat{\boldsymbol{h}})$, and inv does not have fixed point on the set of vertices. Hence, k is even.*

The loop we construct is

$$\hat{\boldsymbol{g}}_{\mathbf{eq}} = \hat{\boldsymbol{h}}|_{u_{k/2} \to u_{k/2+1}} \sqcup \mathrm{inv}(\hat{\boldsymbol{h}})|_{u_{k/2+1} \to u_{k/2}}.$$

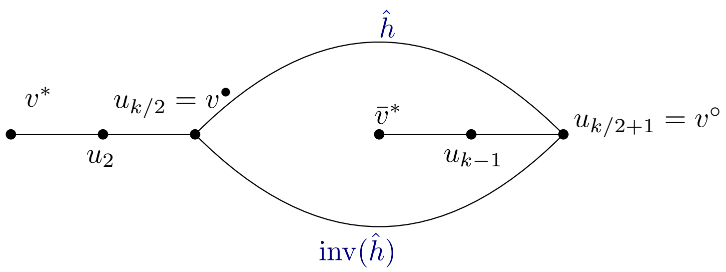


Figure 4.10: Construction of the equilibrium loop.

Definition 4.5.32 (Equilibrium loop). *The loop $\hat{\boldsymbol{g}}_{\mathbf{eq}}$ of Construction 4.5.31 is called the equilibrium loop.*

Let us give some properties of the loop $\hat{\boldsymbol{g}}_{\boldsymbol{eq}}$.

Definition 4.5.33 (Invariant loop)**.** *Let $\boldsymbol{g}$ be a simple loop of even length. We say it is invariant if for all $i \in [\#\boldsymbol{g}]$,*

$$\mathrm{inv}_{\hat{\mathfrak{m}}}\left(\mathrm{vert}(g_{2i-1})\right) = \mathrm{vert}(g_{\#\boldsymbol{g}+2i-1}).$$

Lemma 4.5.34. *The loop equilibrium loop $\hat{\boldsymbol{g}}_{\boldsymbol{eq}}$ is a simple, invariant, good loop.*

Proof. The fact that $\hat{\boldsymbol{g}}_{\boldsymbol{eq}}$ is simple follows by construction: both $\hat{\boldsymbol{h}}$ and $\mathrm{inv}(\hat{\boldsymbol{h}})$ are simple paths as they are geodesics. Furthermore, we chose the paths so that the only vertices that are both in $\hat{\boldsymbol{h}}|_{u_{k/2}\to u_{k/2+1}}$ and $\mathrm{inv}(\hat{\boldsymbol{h}})|_{u_{k/2+1}\to u_{k/2}}$ are their endpoints.

The fact that $\hat{\boldsymbol{g}}_{\mathbf{eq}}$ is invariant also follows by construction: $\mathrm{inv}(\hat{\boldsymbol{h}})|_{u_{k/2+1}\to u_{k/2}}$ is obtained from $\hat{\boldsymbol{h}}|_{u_{k/2}\to u_{k/2+1}}$ by applying inv to each of its half-edges.

Finally, the fact that $\hat{\boldsymbol{g}}_{\mathbf{eq}}$ is good follows from the fact that

$$\hat{\ell}(u_{k/2}) = \hat{\ell}\circ \mathrm{inv}(u_{k/2}) = \hat{\ell}(u_{k/2+1}),$$

and from the fact that $\hat{\boldsymbol{h}}$ and $\mathrm{inv}(\hat{\boldsymbol{h}})$ are geodesics: the maximum of their labels may be attained only once, or twice at consecutive vertices. □

By the Jordan curve Theorem, $\hat{\boldsymbol{g}}_{\mathbf{eq}}$ separates $\hat{\mathfrak{m}}$ into two embedded maps, whose boundary is $\hat{\boldsymbol{g}}_{\mathbf{eq}}$. We denote by $\hat{\mathfrak{m}}_+$ the embedded map containing the face incident to the global root, and by $\hat{\mathfrak{m}}_-$ the other embedded map.

Lemma 4.5.35. *Let $v^\bullet = u_{k/2}$ and $v^\circ = u_{k/2+1}$ in Construction 4.5.31. Let $v_1,\dots,v_d$ the neighbors of v° in $\hat{\mathfrak{m}}_+$. We have for all $i\in[d]$ that*

$$\hat{\ell}(v_i) \geq \hat{\ell}(v^\circ).$$

Proof. Assume that there exists v' in $\hat{\mathfrak{m}}_+$, a neighbor of v°, with $\hat{\ell}(v') = \hat{\ell}(v^\circ) - 1$. The only minima of vertex labels in $\hat{\mathfrak{m}}$ are attained at v^* and $\bar{v}^*$. Hence, there is a geodesic with strictly decreasing vertex labels, of length $\hat{\ell}(v')$, from v' to one of v^* or $\bar{v}^*$. Since the labels are strictly decreasing, the geodesic may not cress the curve $\hat{\boldsymbol{g}}_{\mathbf{eq}}$, so the geodesic goes to v^*. Hence, we can construct a path of length $\ell(v^\circ)$ to v^*. This contradicts the fact that $\hat{\boldsymbol{h}}$ is a geodesic. □

Let us now take $(\hat{\mathfrak{m}},\hat{\ell})$ to be the glued map, constructed from $(\tilde{\mathfrak{m}},\tilde{\ell})$ and the leftmost good geodesic $\tilde{\boldsymbol{g}}$ as defined in Section 4.5.5. The path $\tilde{\boldsymbol{g}}$ corresponds to a good loop $\hat{\boldsymbol{g}}$ in $\hat{\mathfrak{m}}$. We may identify edges of $\tilde{\mathfrak{m}}$ that are not on $\tilde{\boldsymbol{g}}$, to edges in the embedded map $\hat{\mathfrak{m}}_+$. In particular,

$$\tilde{\boldsymbol{h}}|_{v^*\to v^\bullet}$$

can be seen as a path in $\hat{\mathfrak{m}}_+$. The path $\hat{\boldsymbol{g}}$ can be written $\hat{\boldsymbol{g}} = \hat{\boldsymbol{g}}_{\mathbf{1}} \sqcup \hat{\boldsymbol{g}}_{\mathbf{2}}$, with $\hat{\boldsymbol{g}}_{\mathbf{1}}$ starting at $v^\bullet$. We can thus see $\tilde{\boldsymbol{h}}$ as being embedded in $\hat{\mathfrak{m}}_+$:

$$\tilde{\boldsymbol{h}} = \tilde{\boldsymbol{h}}|_{v^*\to v^\bullet} \sqcup \hat{\boldsymbol{g}}_{\mathbf{1}} \quad \text{in } \hat{\mathfrak{m}}_+.$$

Proposition 4.5.36. *The loop $\hat{\boldsymbol{g}}$ is the equilibrium loop $\hat{\boldsymbol{g}}_{\mathbf{eq}}$ of $\hat{\mathfrak{m}}$.*

The proof relies on the following lemma concerning $\tilde{\boldsymbol{h}}$.

Lemma 4.5.37. *The path $\tilde{\boldsymbol{h}}$ is the leftmost geodesic in $\hat{\mathfrak{m}}$ between v^* and v°.*

For the proofs of both Lemma 4.5.37 and Proposition 4.5.36, we use the following paths. Given a vertex u in $\hat{\boldsymbol{g}}$, we set

$$\hat{\boldsymbol{g}}_{\boldsymbol{u}-} = \begin{cases} \hat{\boldsymbol{g}}_{\mathbf{1}}|_{v^\bullet\to u} & \text{if } u \text{ is in } \hat{\boldsymbol{g}}_{\mathbf{1}} \\ \hat{\boldsymbol{g}}_{\mathbf{2}}^{-1}|_{v^\bullet\to u} & \text{if } u \text{ is in } \hat{\boldsymbol{g}}_{\mathbf{2}}, \end{cases} \quad \text{and} \quad \hat{\boldsymbol{g}}_{\boldsymbol{u'}+} = \begin{cases} \hat{\boldsymbol{g}}_{\mathbf{1}}|_{u\to v^\circ} & \text{if } u \text{ is in } \hat{\boldsymbol{g}}_{\mathbf{1}} \\ \hat{\boldsymbol{g}}_{\mathbf{2}}^{-1}|_{u\to v^\circ} & \text{if } u \text{ is in } \hat{\boldsymbol{g}}_{\mathbf{2}}. \end{cases}$$

Note that $\#\hat{\boldsymbol{g}}_{\boldsymbol{u'}-} + \#\hat{\boldsymbol{g}}_{\boldsymbol{u'}-} = \#\hat{\boldsymbol{g}}_{\mathbf{1}} = \#\hat{\boldsymbol{g}}_{\mathbf{2}}$.

Proof of Lemma 4.5.37. Notice first that $\tilde{\boldsymbol{h}}$ is the leftmost geodesic in $\hat{\mathfrak{m}}_+$ between v^* and v°. Let $\tilde{\boldsymbol{h}}'$ be the leftmost good geodesic in $\hat{\mathfrak{m}}$ between v^* and v°. We assume that $\tilde{\boldsymbol{h}} \neq \tilde{\boldsymbol{h}}'$. We show that in that case, we can construct a path in $\hat{\mathfrak{m}}_+$ that is either shorter or to the left of $\tilde{\boldsymbol{h}}$. This will contradict the fact that $\tilde{\boldsymbol{h}}$ is the leftmost good geodesic.

Let u be the last vertex such that

$$\tilde{\boldsymbol{h}}|_{v^*\to u} = \tilde{\boldsymbol{h}}'|_{v^*\to u},$$

and u' be the first vertex strictly after u along $\tilde{\boldsymbol{h}}'$ that is in $\hat{\boldsymbol{g}}$. Since we assumed that $\tilde{\boldsymbol{h}} \neq \tilde{\boldsymbol{h}}'$, we have that $u \neq v^\circ$ and u' is well-defined.

We consider the path $\tilde{\boldsymbol{h}}'|_{v^*\to u'}$. Since $\tilde{\boldsymbol{h}}'$ is a geodesic,

$$\#\tilde{\boldsymbol{h}}'|_{v^*\to u'} \leq \# \left(\tilde{\boldsymbol{h}}|_{v^*\to v^\bullet} \sqcup \hat{\boldsymbol{g}}_{\boldsymbol{u'}-} \right). \tag{4.18}$$

There are two situations, depending on whether $\tilde{\boldsymbol{h}}'|_{u\to u'}$ is contained in $\hat{\mathfrak{m}}_+$ or $\hat{\mathfrak{m}}_-$. Assume we are in the former case, as depicted in Figure 4.11.

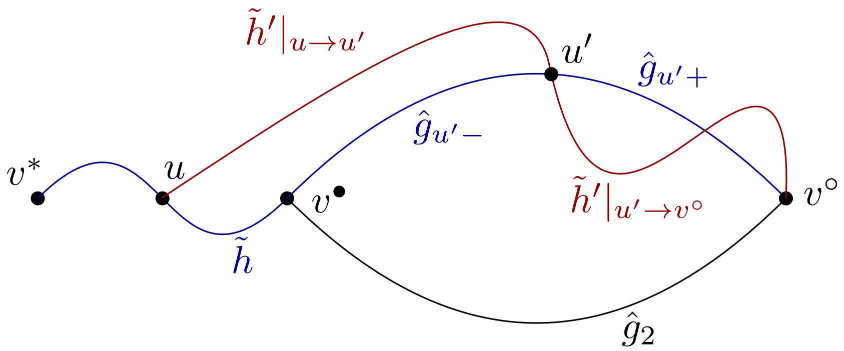


Figure 4.11: The case where $\tilde{\boldsymbol{h}}'|_{u\to u'}$ is in $\hat{\mathfrak{m}}_+$.

If the inequality (4.18) is strict, then

$$\tilde{\boldsymbol{h}}'|_{v^*\to u'} \sqcup \hat{\boldsymbol{g}}_{\boldsymbol{u'}+}$$

is strictly shorter than $\tilde{\boldsymbol{h}}$, this contradict the fact that $\tilde{\boldsymbol{h}}$ is a geodesic in $\hat{\mathfrak{m}}_+$. Thus, we assume there is equality in (4.18). If $\tilde{\boldsymbol{h}}'|_{v^*\to u'}$ is at the left of $\tilde{\boldsymbol{h}}$, then

$$\tilde{\boldsymbol{h}}'|_{v^*\to u'} \sqcup \hat{\boldsymbol{g}}_{\boldsymbol{u'}+}$$

is a geodesic form v^* to v°, contained in $\hat{\mathfrak{m}}_+$, at the left of $\tilde{\boldsymbol{h}}$. This is a contradiction. If $\tilde{\boldsymbol{h}}'|_{v^*\to u'}$ is at the right of $\tilde{\boldsymbol{h}}$, then

$$\tilde{\boldsymbol{h}}^{(1)} = \tilde{\boldsymbol{h}}|_{v^*\to v^\bullet} \sqcup \hat{\boldsymbol{g}}_{\boldsymbol{u'}-} \sqcup \tilde{\boldsymbol{h}}'|_{u'\to v^\circ}$$

is a geodesic from v^* to v°, at the left of $\tilde{h}'$. This contradicts the fact that $\tilde{\boldsymbol{h}}'$ is the leftmost geodesic from v^* to v°. Thus, $\tilde{\boldsymbol{h}}'|_{u\to u'}$ cannot be in $\hat{\mathfrak{m}}_+$.

We now assume that $\tilde{\boldsymbol{h}}'|_{u\to u'}$ is in $\hat{\mathfrak{m}}_-$. It implies in particular that u is in $\hat{\boldsymbol{g}}_{\boldsymbol{1}}$. This situation is depicted in Figures 4.12 and 4.13.

If (4.18) is strict, then

$$\tilde{\boldsymbol{h}}^{(2)} = \tilde{\boldsymbol{h}}|_{v^*\to v^\bullet} \sqcup \mathrm{inv}\left(\tilde{\boldsymbol{h}}'|_{v^\bullet\to u'} \sqcup \hat{\boldsymbol{g}}_{\boldsymbol{u'}+} \right)$$

is a path from v^* to v°, contained in $\hat{\mathfrak{m}}_+$, and strictly shorter than $\tilde{\boldsymbol{h}}$. This contradicts the fact that $\tilde{\boldsymbol{h}}$ is a geodesic in $\hat{\mathfrak{m}}_+$. Hence, (4.18) is an equality. In that case, if u' is in $\hat{\boldsymbol{g}}_{\boldsymbol{1}}$, we see that

$$\#\tilde{\boldsymbol{h}}|_{v^*\to u'} = \#\tilde{\boldsymbol{h}}'|_{v^*\to u'},$$

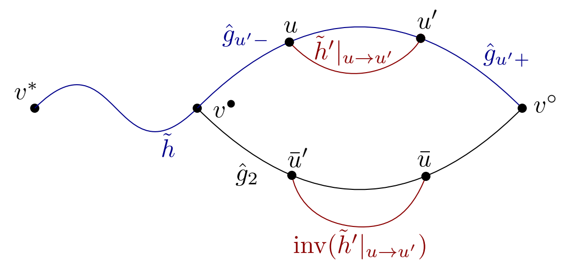


Figure 4.12: The case where $\tilde{\boldsymbol{h}}'|_{u\to u'}$ is in $\hat{\mathfrak{m}}_-$, and u' is in $\hat{\boldsymbol{g}}_1$.

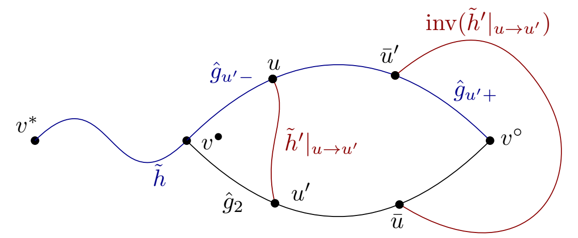


Figure 4.13: The case where $\tilde{\boldsymbol{h}}'|_{u\to u'}$ is in $\hat{\mathfrak{m}}_-$, and u' is in $\hat{\boldsymbol{g}}_2$.

and that $\tilde{\boldsymbol{h}}$ is at the left of $\tilde{\boldsymbol{h}}'$ at u. We can thus construct a path of the same length as $\tilde{\boldsymbol{h}}'$ that is to the left of $\tilde{\boldsymbol{h}}'$. This is a contradiction. If u' is in $\hat{\boldsymbol{g}}_2$, then $\tilde{\boldsymbol{h}}^{(2)}$ is at the left of $\tilde{\boldsymbol{h}}$, is of the same length, and is contained in $\hat{\mathfrak{m}}_+$. This contradicts the fact that $\tilde{\boldsymbol{h}}$ is the leftmost geodesic from v^* to v°.

We thus necessarily have that $\tilde{\boldsymbol{h}} = \tilde{\boldsymbol{h}}'$. □

Proof of Proposition 4.5.36. Lemma 4.5.37 implies that $\tilde{\boldsymbol{h}}$ is the leftmost geodesic from the root vertex v^* to v°.

Consider now the leftmost geodesic $\hat{\boldsymbol{h}}$ from v^* to $\bar{v}^* = \mathrm{inv}(v^*)$. Let u be the last vertex such that

$$\tilde{\boldsymbol{h}}|_{v^*\to u} = \hat{\boldsymbol{h}}|_{v^*\to u}.$$

Assume that $u \neq v^\circ$ and let $u' \neq u$ be the last vertex along $\hat{\boldsymbol{h}}$ that is both in $\hat{\boldsymbol{g}}$ and $\hat{\boldsymbol{h}}$.

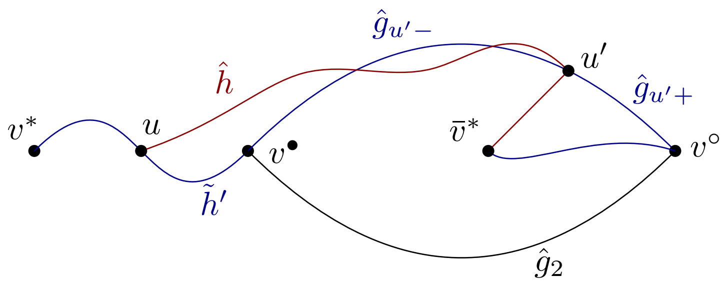


Figure 4.14: The different paths involved in the proof of Proposition 4.5.36.

We define the path

$$\tilde{\boldsymbol{h}}' = \tilde{\boldsymbol{h}} \sqcup \mathrm{inv}(\tilde{\boldsymbol{h}}|_{v^*\to v^\bullet}).$$

This is a path from v^* to $\bar{v}^*$.

We claim that $\tilde{\boldsymbol{h}}'$ is a geodesic from v^* to $\bar{v}^*$. Assuming the claim for now, we observe that $\hat{\boldsymbol{h}}$ is at the left of $\tilde{\boldsymbol{h}}'$. Thus, the path

$$\hat{\boldsymbol{h}}' = \hat{\boldsymbol{h}}|_{v^*\to u'} \sqcup \hat{\boldsymbol{g}}_{\boldsymbol{u}'+}$$

is at the left of $\tilde{\boldsymbol{h}}'$ as well. However,

$$\#\hat{\boldsymbol{h}}' = \#\hat{\boldsymbol{h}}|_{v^*\to u'} + \#\hat{\boldsymbol{g}}_{\boldsymbol{u}'+} \leq \# \left(\tilde{\boldsymbol{h}}|_{v^*\to v^\bullet} \sqcup \hat{\boldsymbol{g}}_{\boldsymbol{u}'-}\right) + \#\hat{\boldsymbol{g}}_{\boldsymbol{u}'+} = \#\tilde{\boldsymbol{h}}.$$

Hence, $\hat{\boldsymbol{h}}'$ is a geodesic from v^* to v° on the left of $\tilde{\boldsymbol{h}}$. As $\tilde{\boldsymbol{h}}$ is the leftmost geodesic from v^* to v°, we have $\tilde{\boldsymbol{h}} = \hat{\boldsymbol{h}}'$. This implies that $\tilde{\boldsymbol{h}}|_{v^*\to u'} = \hat{\boldsymbol{h}}|_{v^*\to u'}$. This can only be satisfied if $u = u' = v^\circ$, i.e. if $\tilde{\boldsymbol{h}} = \hat{\boldsymbol{h}}|_{v^*\to v^\circ}$. This immediately implies that $v^\bullet$ is part of $\hat{\boldsymbol{h}}$ and

$$\tilde{\boldsymbol{h}}|_{v^\bullet\to v^\circ} = \hat{\boldsymbol{h}}|_{v^\bullet\to v^\circ}.$$

Hence, we deduce that $\hat{\boldsymbol{g}} = \hat{\boldsymbol{g}}_{\mathbf{eq}}$.

We now prove the claim. We proceed in two parts: we first prove that

$$\#\hat{\boldsymbol{h}}|_{v^*\to u'} = \#\left(\tilde{\boldsymbol{h}}|_{v^*\to v^\bullet} \sqcup \hat{\boldsymbol{g}}_{\boldsymbol{u'}-}\right), \tag{4.19}$$

and then that

$$\#\hat{\boldsymbol{h}}|_{u'\to\bar{v}^*} = \#\left(\hat{\boldsymbol{g}}_{\boldsymbol{u'}+} \sqcup \tilde{\boldsymbol{h}}'|_{v^\circ\to\bar{v}^*}\right). \tag{4.20}$$

This will imply the claim:

$$\#\hat{\boldsymbol{h}} = \#\hat{\boldsymbol{h}}|_{v^*\to u'} + \#\hat{\boldsymbol{h}}|_{u'\to\bar{v}^*} = \#\tilde{\boldsymbol{h}}|_{v^*\to v^\bullet} + \underbrace{\#\hat{\boldsymbol{g}}_{\boldsymbol{u'}-} + \#\hat{\boldsymbol{g}}_{\boldsymbol{u'}+}}_{=\#\hat{\boldsymbol{g}}_{\boldsymbol{1}}} + \#\tilde{\boldsymbol{h}}'|_{v^\circ\to\bar{v}^*} = \#\tilde{\boldsymbol{h}}'.$$

To prove (4.19), we notice that since $\hat{\boldsymbol{h}}$ is a geodesic,

$$\#\hat{\boldsymbol{h}}|_{v^*\to u'} \leq \#\left(\tilde{\boldsymbol{h}}|_{v^*\to v^\bullet} \sqcup \hat{\boldsymbol{g}}_{\boldsymbol{u'}-}\right).$$

However, if the inequality were strict, we could construct

$$\hat{\boldsymbol{h}}^{(\boldsymbol{1})} = \hat{\boldsymbol{h}}|_{v^*\to u'} \sqcup \hat{\boldsymbol{g}}_{\boldsymbol{u'}+},$$

a path from $\boldsymbol{v^*}$ to v° that is strictly shorter than $\tilde{\boldsymbol{h}}$. This would contract the fact that $\tilde{\boldsymbol{h}}$ is a geodesic.

To prove (4.20), we proceed similarly: $\hat{\boldsymbol{h}}$ being a geodesic implies

$$\#\hat{\boldsymbol{h}}|_{u'\to\bar{v}^*} \leq \#\left(\hat{\boldsymbol{g}}_{\boldsymbol{u'}+} \sqcup \hat{\boldsymbol{h}}|_{v^\bullet\to\bar{v}^*}\right).$$

If the inequality were strict, the path

$$\hat{\boldsymbol{h}}^{(\boldsymbol{2})} = \mathrm{inv}\left(\hat{\boldsymbol{g}}_{\boldsymbol{u'}-} \sqcup \hat{\boldsymbol{h}}|_{u'\to\bar{v}^*}\right)$$

would be a path from v^* to v° that is strictly shorter than $\tilde{\boldsymbol{h}}$. This conclude the proof of the claim. □

4.5.6 The mapping for maps on $\mathbb{RP}^2$

We can now give the full construction.

Construction 4.5.38. *Consider a suitably labelled map $(\tilde{\mathfrak{m}}, \tilde{\ell})$ with two local minima. Choose a path $\tilde{\boldsymbol{g}}$ as in Section 4.5.5. Construct the glued map $(\hat{\mathfrak{m}}, \hat{\ell})$ as in Section 4.5.4. This map comes equipped with an orientation-reversing matching* inv*. Using the construction of Section 4.5.3, we see $\hat{\mathfrak{m}}$ as a map on the orientation covering of a map $\mathfrak{m}$ on the projective plane $\mathbb{RP}^2$. This map is naturally pointed: the pointed vertex is the common image of the two vertices labelled by* 0*.*

We now give the inverse construction. The idea is as follows: we can choose canonically a loop – the equilibrium loop – in the orientable double covering of $\mathfrak{m}$. We can then contract the loop to obtain a suitably labelled map with two local minima.

Let $\mathfrak{m}$ be a non-orientable map on the projective plane $\mathbb{RP}^2$ which is flag-labelled by $\lambda\colon \mathrm{Fl}_{\mathfrak{m}} \to [n]$ and with vertex profile $\theta\bar{\theta}$. We assume that $\mathfrak{m}$ is pointed, i.e. that there is a distinguished vertex v in $\mathfrak{m}$. We label the vertices of $\mathfrak{m}$ by their geodesic distance to v, giving a suitable labelling ℓ of $\mathfrak{m}$. Using the bijection of Proposition 4.5.8, we construct a half-edge labelled map $\hat{\mathfrak{m}}$, equipped with an orientation-reversing matching $\mathrm{inv} = \rho_{\mathfrak{m}}$. This is the orientation covering map of $\mathfrak{m}$. The labelling of the vertices of $\mathfrak{m}$ induces a labelling of the vertices of $\hat{\mathfrak{m}}$: for every vertex $\hat{v}$ of $\hat{\mathfrak{m}}$, there is a unique image vertex v (by the projection from the orientation covering) in $\mathfrak{m}$. We set

$$\hat{\ell}(\hat{v}) = \ell(v).$$

The map $(\hat{\mathfrak{m}}, \hat{\ell})$ is a suitably labelled map: the minimum of the labels is 0, and the difference between the labels of two vertices connected by an edge is at most one since every edge in $\hat{\mathfrak{m}}$ is in the preimage of an edge in $\mathfrak{m}$. In $\hat{\mathfrak{m}}$, we choose $\hat{\boldsymbol{g}}_{\mathbf{eq}}$ to be the unique equilibrium loop in $\hat{\mathfrak{m}}$ (associated with inv).

Before constructing the map $\tilde{\mathfrak{m}}$, we exchange some of the labels in $\hat{\mathfrak{m}}$.

Construction 4.5.39 (Flipping the labels)**.** *By the Jordan curve theorem, the equilibrium loop $\hat{\boldsymbol{g}}_{\mathbf{eq}}$ separates $\hat{\mathfrak{m}}$ into two embedded maps: $\hat{\mathfrak{m}}_+$ containing the root face, and $\hat{\mathfrak{m}}_-$ the other one. Each face in $\hat{\mathfrak{m}}$ corresponds to a cycle of θ or $\bar{\theta}$. All the labels of a face are either in $[n]$ in the first case or in $[\bar{n}]$ in the second case. For each face f in $\hat{\mathfrak{m}}_+$, if the labels of the half-edges incident to f (i.e. such that f is at their left) are in $[\bar{n}]$, exchange their label with the one obtained by applying* inv*. Similarly, change the labels of the half-edges in a face of $\hat{\mathfrak{m}}_-$ with the one obtained by applying* inv *if they are in $[n]$. This mapping that flips the labels is $2^{c(\theta)-1}$-to-1, as each pair of faces in $\hat{\mathfrak{m}}$, except the one of the root face, may be flipped.*

Once the labels are exchanged we can glue $\hat{\boldsymbol{g}}_{\mathbf{eq}}$ to itself to obtain a suitably labelled map with two local minima.

Construction 4.5.40 (Closing the slit)**.** *Consider now the map $\hat{\mathfrak{m}}_+$ with boundary $\hat{\boldsymbol{g}}$, embedded in $\hat{\mathfrak{m}}$. We glue the boundary to itself to remove the boundary face. The good loop $\hat{\boldsymbol{g}}$ can be written as the concatenation of two good paths*

$$\hat{\boldsymbol{g}} = \hat{\boldsymbol{g}}_{\mathbf{1}} \sqcup \hat{\boldsymbol{g}}_{\mathbf{2}}.$$

Let $v^\bullet$ and v° be the first and last vertex of $\hat{\boldsymbol{g}}_{\mathbf{1}}$, they are the two minima of these good paths. We glue the two paths together, identifying the vertices as follows: for each $i = 1, \dots, d-1$ there are exactly two vertices at distance i to $v^\bullet$, we identify them. Denote by $\tilde{\mathfrak{m}}$ the resulting map.

Note that thanks to Construction 4.5.39 The face permutations of $\tilde{\mathfrak{m}}$ is θ.

Lemma 4.5.41. *The resulting map $\tilde{\mathfrak{m}}$ has exactly two local minima.*

Proof. In $\hat{\mathfrak{m}}^+$ there may be local minima only at the root, or on the boundary of $\hat{\mathfrak{m}}^+$. The two possible vertices where there might be minima are $v^\bullet$ and v°. However, by construction, one of them, say $v^\bullet$, is connected to the root and may not be a local minimum. Lemma 4.5.35, however, implies that v° is a local minimum in $\tilde{\mathfrak{m}}$, as the vertices that may be of lower label that it get removed in Construction 4.5.40. □

We can now state the main theorem of this section.

Theorem 4.5.42. *The previous construction gives a $2^{c(\theta)-1}$-to-1 mapping between the set of pointed labelled maps on the projective plane with face profile given by $\theta\bar{\theta}$, and the set of suitably labelled maps with two local minima and face profile θ.*

Proof. Denote by Φ this mapping. Lemma 4.5.41 implies that this construction gives a well-defined map to the set of suitably labelled map with two local minima. The fact that the face profile remains θ is a consequence of the construction. In particular, the faces are not modified during the cutting and gluing. The mapping Φ can be seen as the composition of a mapping Φ_1, $2^{c(\theta)-1}$-to-1, that flips the labels, see Construction 4.5.39, and a bijection Φ_2 that consists in cutting the orientable double cover and gluing the sides appropriately. The fact that Φ_2 is a bijection follows from Propositions 4.5.8, Proposition 4.5.22, and the fact that given an orientable double cover, there is a unique equilibrium loop by Construction 4.5.31, along which we cut. This is the same curve we construct in the reverse construction by Proposition 4.5.36. □

Theorem 4.5.42 allows us to conclude the proof of Corollary 4.1.3.

Proof of Corollary 4.1.3. We proved in Proposition 4.4.2 that:

$$N^{l-2-n/2}\kappa_l(\boldsymbol{n}) = \left(\frac{2}{\beta}\right)^{l-1} \#\mathcal{M}_0(\theta(\boldsymbol{n})) + \left(\frac{2}{\beta}\right)^{l-1} \frac{1}{N}\left(\frac{2}{\beta} - 1\right)\frac{\#\mathcal{S}_2(\theta(\boldsymbol{n}))}{n/2 - l + 1} + \mathcal{O}\left(\frac{1}{N^2}\right).$$

Denote by $\mathcal{M}_{1/2}(\theta(\boldsymbol{n}))$ the set of edge-labelled maps on $\mathbb{RP}^2$ with face profile $\theta(\boldsymbol{n})$. Theorem 4.5.42 implies:

$$(1 + n/2 - l)\,\#\mathcal{M}_{1/2}(\theta(\boldsymbol{n})) = 2^{l-1}\#\mathcal{S}_2(\theta(\boldsymbol{n})),$$

as $(1+n/2-l)$ is the number of choice of a marked vertex in a map on $\mathbb{RP}^2$ with l faces and $n/2$ edges. Hence,

$$N^{l-2-n/2}\kappa_l(\boldsymbol{n}) = \left(\frac{2}{\beta}\right)^{l-1} \#\mathcal{M}_0(\theta(\boldsymbol{n})) + \left(\frac{2}{\beta}\right)^{l-1} \frac{1}{N}\left(\frac{2}{\beta}-1\right)\#\mathcal{M}_1(\theta(\boldsymbol{n})) + \mathcal{O}\left(\frac{1}{N^2}\right),$$

which is the wanted result. ☐

4.6 The limit $\beta \to \infty$ and the roots of Hermite polynomials

We give a simple application of Theorem 4.1.2. Let us take the limit $\beta \to \infty$ in (4.15). We get for all $n \geq 2$ even,

$$\kappa_1(n) = \sum_{q+r+s=n/2-l+1} \frac{(-1)^q B_r}{s+1}\binom{r+s}{r} N^{s+1-n/2}\left\langle e_q\right\rangle_{\theta,l-1}. \tag{4.21}$$

The terms of the expansion are linear combinations of expectations of product of distances in planar maps with one face of degree n: with Remark 4.4.1 in mind, we see that since $\#V_{\mathfrak{m}}^{\min} \geq 1$, we have $l + \#V_{\mathfrak{m}}^{\min} - (l-1) \geq 2$. Necessarily, $\#V_{\mathfrak{m}}^{\min} = 1$ and the genus must be zero.

If we take $l=1$, the case of trees, we can push the computation further. We have that

$$T_\infty^N := \lim_{\beta\to\infty} T_\beta^N = \frac{1}{\sqrt{N}}\begin{pmatrix} 0 & \sqrt{N-1} & 0 & 0 & 0 & \dots \\ \sqrt{N-1} & 0 & \sqrt{N-2} & 0 & 0 & \dots \\ 0 & \sqrt{N-2} & 0 & \ddots & \ddots & \dots \\ \vdots & \ddots & \ddots & \ddots & \sqrt{2} & 0 \\ 0 & \dots & 0 & \sqrt{2} & 0 & \sqrt{1} \\ 0 & \dots & 0 & 0 & \sqrt{1} & 0 \end{pmatrix}. \tag{4.22}$$

The eigenvalues of $\sqrt{N}T_\infty^N$ are the roots of the Hermite polynomials He_N defined by

$$\mathrm{He}_N(x) := (-1)^N \mathrm{e}^{x^2/2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^N \mathrm{e}^{-x^2/2} \text{ for } N \geq 1. \tag{4.23}$$

In particular, if we denote by $h_{1,N} \leq h_{2,N} \leq \dots \leq h_{N,N}$ the roots of He_N. We have that

$$p_n(h_{1,N},\dots,h_{N,N}) = \kappa_1(n), \tag{4.24}$$

for all $N \geq 1$. This is easily seen from the fact that the characteristic polynomial of T_∞^N satisfies

$$\begin{cases} \det\left(z - T_\infty^1\right) = z \\ \det\left(z - T_\infty^{N+1}\right) = z\det\left(z - T_\infty^N\right) - N\det\left(z - T_\infty^{N-1}\right), \end{cases}$$

the same induction equations as for $(\mathrm{He}_N)_{N\geq 1}$ (see for instance [AGZ10, Section 3.2.2] for many properties of the Hermite polynomials).

We give one application of (4.24). The two leading orders of $p_n(h_{1,N},\dots,h_{N,N})$ are given by [KM16]:

$$N^{-n/2-1}p_n(h_{1,N},\dots,h_{N,N}) = \mathrm{Cat}_{n/2} - \frac{1}{N}\left(2^{n-1} - \binom{n-1}{n/2}\right) + \mathcal{O}\left(\frac{1}{N^2}\right).$$

We recover that the number of planar trees with $n/2$ edges is the Catalan number $\mathrm{Cat}_{n/2}$, and obtain that

$$\#\mathcal{S}_2((1\,2\,\dots\,n)) = \frac{n}{2}\left(2^{n-1} - \binom{n-1}{n/2}\right),$$

or equivalently

$$\sum_{q+r=1} \frac{(-1)^q B_r}{n/2} \binom{r+n/2-1}{r} \langle e_q \rangle_{\theta,0} = \frac{1}{2} \langle 1 \rangle_{\theta,0} - \frac{2}{n} \langle d \rangle_{\theta,0} = -\left(2^{n-1} - \binom{n-1}{n/2}\right),$$

that is:

$$\frac{\langle d \rangle_{\theta,0}}{\langle 1 \rangle_{\theta,0}} = \frac{2^{n-2} n}{\mathrm{Cat}_{n/2}(n/2+1)} - \frac{n^2}{8(n/2+1)} \sim \frac{1}{2}\sqrt{\frac{\pi}{8}} n^{3/2} \text{ as } n \to \infty.$$

The above quantity is the average distance between two (distinguished) uniform vertices in a tree. This is related to the expectation of the area under a Brownian excursion $\mathbb{E}\mathcal{B}_{\mathrm{ex}} = \sqrt{\pi/8}$, see [Jan07].

Chapter 5

Fay identities of Pfaffian type for Hyperelliptic curves

This Chapter is based on a joint work with Gaëtan Borot [BB24].

5.1 Introduction

We saw in Section 2.7 that Fay's identity (Theorem 2.7.2) is not only a geometric identity, but also one of many links between the geometry of Riemann surfaces and the theory of integrable systems. On the other hand, the integrable β-ensemble, for $\beta \in \{1, 2, 4\}$ are also related to integrable systems. They admit many determinantal ($\beta = 2$) and Pfaffian ($\beta = 1, 4$) formulae involving observables such as expectation values of ratios of characteristic polynomials, as discussed in Section 2.7.4.

The large N asymptotic of these matrix models has been extensively studied, either by Riemann–Hilbert methods relying on integrability [Pas06; CGM15; Cha+23], or by probabilistic techniques [Joh98; APS01; BG13; BG24; Shc13; BGK14]. In the one-cut regime, i.e. when the large N spectral density of the random matrix is supported in a segment, the asymptotics are described by an asymptotic expansion in $1/N$. In particular, for $\beta = 2$, the terms of the asymptotic expansion are generating series of maps (see Section 2.8.1). In the multi-cut regime, when the large N spectral density is supported on $g + 1$ disjoint segments, a phenomenon of "tunneling" of the eigenvalues appear and the asymptotic expansion include oscillatory terms, modeled by theta functions (see Section 2.8.2). This theta function is naturally the Riemann theta function of the spectral curve (see Section 2.7.1) of the β-model, which is hyperelliptic of genus g. The exact determinantal identities for $\beta = 2$ allow us to recover in the large N limit the Fay identities. This implication will be shown in Proposition 5.5.2.

The purpose of our work is to generalize this to orthogonal and quaternionic self-dual 1-matrix models. The determinantal formulae of the hermitian case for $2n$-point functions of ratios of characteristic polynomials are then replaced with the Pfaffian formulae found by Borodin and Strahov [BS06]. These models correspond to the $\beta = 1$ and $\beta = 4$ cases of the β-ensembles, whose asymptotic analysis in the $(g + 1)$-cut regime has been established for all $\beta > 0$ by probabilistic techniques in [BGK14; Shc13; BG24]. The large N spectral density is described by a hyperelliptic curve of genus g independent of β and having a period matrix $\boldsymbol{\tau}$. The asymptotics of the partition function and the $2n$-point functions are governed by the theta function associated with the matrix $\frac{\beta}{2}\boldsymbol{\tau}$. Inserting these asymptotics up to $o(1)$ in the Pfaffian identities for the $2n$-point functions yield identities between these theta functions, which can be expressed solely in terms of the geometry of the underlying spectral curve.

The spectral curves arising from the large N limit of the matrix models we consider must be hyperelliptic, have real Weierstraß points, and have the Boutroux property. We show that all such curves can be realized as the spectral curve of an off-critical β-ensemble with polynomial potential (Proposition 5.3.11). By analytic continuation we can extend the validity of the resulting identities to all hyperelliptic curves. This gives our main results: Theorem 5.5.1 for $\beta = 2$, Theorem 5.5.4 for $\beta = 1$ and Theorem 5.5.7 for $\beta = 4$. It turns out that all three identities can be reformulated in terms of theta functions for the matrix of periods $\boldsymbol{\tau}$ (instead of $\frac{\beta}{2}\boldsymbol{\tau}$) and in this form we are able to give them a second proof by direct algebraic methods. Interestingly, the $\beta = 1$ and $\beta = 4$ identities are equivalent via the modular properties of theta functions, and the $\beta = 2$ identity implies the Fay identity in the special case of hyperelliptic curves.

As a byproduct of our proofs, we obtain a seemingly new formula (Proposition 5.4.3 proved in Section 5.5.4) for the equilibrium energy of the β-ensembles in the multi-cut regime in terms of the geometry of the spectral curve. Although the ingredients are the same, at first sight it does not have exactly the same form as the 1-matrix model specialization of the formula known in the context of the 2-matrix model [Ber03]. Independently of our analysis, we also establish (Proposition 5.4.4 proved in Section 5.6) an explicit formula for the derivative with respect to filling fractions of the equilibrium entropy. For $\beta \neq 2$, the equilibrium entropy appears as the order N term in the free energy of the β-ensemble, and its derivatives with respect to filling fractions appear both in the asymptotics of the partition function (Theorem 5.4.1) and in the centering in the generalized central limit theorem (Theorem 5.4.2).

The strategy of proof via asymptotics in integrable random matrix ensembles is somehow more interesting than the resulting identities in the particular case we studied, and constitutes the originality of this study. In principle this strategy can be applied to any random matrix ensemble:

(i) which is amenable to asymptotic analysis up to $o(1)$ in the large size limit and in the multi-cut regime;

(ii) in which exact formulae for $2n$-point functions in terms of k-point functions (with k independent of n) are available.

Finally, we remark that similar formulae can be derived from the study of quasiperiodic solutions to the Pfaff-Toda Lattice [ASM02]. It is however unclear how these formulae are related to ours.

5.2 β-ensembles and their properties

We recall a few facts about the β-ensembles and review the determinantal and pfaffian formulae of Borodin and Strahov [BS06]. We use the notation $[g]$ for the integer set $\{1, \ldots, g\}$.

5.2.1 The unconstrained model

Fix a finite union A of compact intervals of $\mathbb{R}$, a positive integer N, a real number $\beta > 0$, and an even-degree polynomial V (the potential) with real coefficients and positive top coefficient. We consider the probability measure $\mathbb{P}_N^V$ on A^N defined by

$$\mathrm{d}\mathbb{P}_N^V(\boldsymbol{\lambda}) = \frac{1}{Z_N^V}\,|\Delta(\boldsymbol{\lambda})|^{\beta}\mathrm{e}^{-\frac{\beta N}{2}\sum_{i=1}^N V(\lambda_i)}\prod_{i=1}^N \mathbb{1}_A(\lambda_i)\mathrm{d}\lambda_i, \tag{5.1}$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N) \in A^N$, $\Delta(\boldsymbol{\lambda}) = \prod_{i<j}(\lambda_j - \lambda_i)$ is the Vandermonde determinant, and

$$Z_N^V = \int_{A^N} |\Delta(\boldsymbol{\lambda})|^{\beta} \exp\Big(-\frac{\beta N}{2}\sum_{i=1}^N V(\lambda_i)\Big)\prod_{i=1}^N \mathrm{d}\lambda_i$$

is the partition function. Many results that we quote are formulated with $A = \mathbb{R}$ but their validity trivially extend to the case of A compact. We choose to work from the start with a compact A as it facilitates the statement of the asymptotic results we will need and does not lead to any loss of generality.

When $\beta = 1, 2$ or 4, $\mathbb{P}_N^V$ is the distribution of the N eigenvalues of a random matrix whose law is proportional to $e^{-\frac{\beta N}{2}V(\boldsymbol{M})}\mathrm{d}\boldsymbol{M}$, where $\boldsymbol{M}$ is a matrix which is real symmetric ($\beta = 1$), Hermitian

($\beta = 2$) or quaternionic self-dual ($\beta = 4$) and which is conditioned to have spectrum in A. The measure $\mathrm{d}\boldsymbol{M}$ is the product of Lebesgue measure on the $\mathbb{R}$-linearly independent entries of $\boldsymbol{M}$. In particular, when $V(\boldsymbol{M}) = \frac{1}{2}\boldsymbol{M}^2$, the entries $M_{ij}, i \leq j$ of the matrix are independent Gaussian random variables. These matrix ensembles are known under the name of Gaussian Orthogonal Ensemble (GOE) for $\beta = 1$, Gaussian Unitary Ensemble (GUE) for $\beta = 2$, and Gaussian Symplectic Ensemble (GSE) for $\beta = 4$, see [Meh04]. The β-ensembles (5.1) constitute a generalization of these models.

5.2.2 The model with fixed filling fractions

Let us write $A = \bigsqcup_{h=0}^{g} A_h$ where A_h are the connected components of A. In addition to the measure (5.1), we define the β-ensemble with fixed filling fractions as follows. Let $\boldsymbol{N} = (N_h)_{h=1}^{g} \in (\mathbf{N}^*)^g$ such that $N_1 + \cdots + N_g < N$ and introduce $N_0 \in \mathbb{Z}_{>0}$ such that

$$N_0 + \cdots + N_g = N.$$

We call N_h/N the filling fraction of A_h. We define the measure with fixed filling fractions by

$$\mathrm{d}\mathbb{P}^V_{N,\boldsymbol{N}/N} = \frac{1}{Z^V_{N,\boldsymbol{N}/N}} |\Delta(\boldsymbol{\lambda})|^\beta \exp\bigg(-\frac{\beta N}{2}\sum_{h=0}^{g}\sum_{i=1}^{N_h} V(\lambda_{h,i})\bigg) \prod_{h=0}^{g}\prod_{i=1}^{N_h} \mathbb{1}_{A_h}(\lambda_{h,i})\mathrm{d}\lambda_{h,i}, \tag{5.2}$$

where $\boldsymbol{\lambda} = (\lambda_{h,1},\ldots,\lambda_{h,N_h})_{h=0}^{g}$ is a N-tuple and

$$Z^V_{N,\boldsymbol{N}/N} = \int_{A^N} |\Delta(\boldsymbol{\lambda})|^\beta \mathrm{e}^{-\frac{\beta N}{2}\sum_{h=0}^{g}\sum_{i=1}^{N_h} V(\lambda_{h,i})} \prod_{h=0}^{g}\prod_{i=1}^{N_h} \mathbb{1}_{A_h}(\lambda_{h,i})\mathrm{d}\lambda_{h,i}$$

is the partition function for fixed filling fractions. To distinguish it from the model with fixed filling fractions, we refer to (5.1) as the unconstrained model.

5.2.3 Equilibrium measures and their Stieltjes transform

We define the empirical measure as $L_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}$. It belongs to the space of probability measures on A, which we equip with the weak topology. We first consider L_N in the original model. The following result comes large deviation arguments [AGZ10, Theorem 2.6.1 and Corollary 2.6.3], but see also [Dei00; APS01; Joh98].

Theorem 5.2.1. *Assume that V is an even-degree real polynomial with positive top coefficient. As $N \to \infty$, L_N converges under $\mathbb{P}^V_N$ almost surely, and in expectation (when tested against continuous bounded functions) to the unique probability measure μ_{eq} on A maximising the energy*

$$\mathcal{E}[\mu] = \frac{\beta}{2}\iint_{A^2}\bigg(\ln|\xi-\eta| - \frac{V(\xi)+V(\eta)}{2}\bigg)\mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta). \tag{5.3}$$

Furthermore, μ_{eq} has compact support S consisting in a finite union of segments. It is characterised by the existence of a constant c such that

$$\forall x \in A \qquad 2\int_A \ln|x-\xi|\mathrm{d}\mu_{eq}(\xi) - V(x) \leq c, \tag{5.4}$$

with equality μ_{eq}-almost everywhere.

We will only need to consider $S = \bigsqcup_{h=0}^{g} S_h$ where S_h is a segment contained in the interior[1] $\mathring{A}_h$ of A_h. Without loss of generality one can and one will restrict A_h to be a small enlargement of S_h. The choice of this enlargement will be irrelevant for our purposes, as it does not change the equilibrium measure and only affects the model by corrections which are exponentially small in N, see e.g. [APS01, Proposition 2] or the discussion in [BG13, Section 2] and references therein. In the model with fixed filling fractions, Theorem 5.2.1 has the following adaptation.

[1] This is usually called the 'soft edge' case, by opposition to hard edges that are endpoints of S in the boundary of A.

Theorem 5.2.2. *[BG24, Theorem 1.2] Consider a sequence indexed by N of g-tuples of nonnegative integers $\boldsymbol{N} = (N_1, \dots, N_g)$ with $\sum_{h=1}^{g} N_h < N$ and assume there exists $\boldsymbol{\epsilon} = (\epsilon_h)_{1=0}^{g}$ such that $N_h/N \to \epsilon_h$ for all $h \in [g]$. Then, $L_N = \frac{1}{N}\sum_{h=0}^{g}\sum_{i=1}^{N_h}\delta_{\lambda_{h,i}}$ converges almost surely and in expectation under $\mathbb{P}^V_{N,\boldsymbol{N}/N}$ towards a deterministic probability measure $\mu_{eq,\boldsymbol{\epsilon}}$, which is the maximiser of (5.3) among probability measures giving mass ϵ_h to the segment A_h for each $h \in [0, g]$. It is characterised by the existence of constants $(c_h)_{h=0}^{g}$ such that*

$$\forall h \in [0, g] \quad \forall x \in A_h \qquad 2\int_A \ln|x - \xi| \mathrm{d}\mu_{eq,\boldsymbol{\epsilon}}(\xi) - V(x) \le c_h$$

with equality $\mu_{eq,\boldsymbol{\epsilon}}|_{A_h}$-almost everywhere.

The filling fractions at equilibrium $\boldsymbol{\epsilon}^* = (\epsilon_h^*)_{h=1}^{g}$ are defined as $\epsilon_h^* = \mu_{\mathrm{eq}}(A_h)$, and one can show that $\mu_{\mathrm{eq}} = \mu_{\mathrm{eq},\boldsymbol{\epsilon}^*}$ — see [BG24, Section 1.4].

Let us now discuss the properties of the equilibrium measure, both in the unconstrained case (Theorem 5.2.1) or fixed filling fraction case (Theorem 5.2.2) . We introduce the Stieltjes transform of the equilibrium measure

$$W_1(x) = \int_A \frac{\mathrm{d}\mu_{\mathrm{eq}}(\xi)}{x - \xi},$$

defined for $x \in \mathbb{C} \setminus S$. In [Joh98], Johansson introduces the polynomial

$$P(x) = \int_A \frac{V'(x) - V'(\xi)}{x - \xi} \mathrm{d}\mu_{\mathrm{eq}}(\xi),$$

and derives the equation

$$W_1(x)^2 - V'(x)W_1(x) + P(x) = 0, \tag{5.5}$$

for all $x \in \mathbb{C} \setminus S$. This equation is the large N limit of the first Dyson–Schwinger equation of the model, and its origin can be traced back to [Bré+78; Mig83]. In particular, it implies that

$$W_1(x) = \frac{V'(x)}{2} \pm \frac{\sqrt{V'(x)^2 - 4P(x)}}{2}. \tag{5.6}$$

The determination of the square root should be chosen such that $W_1(x) \sim \frac{1}{x}$ as $x \to \infty$ and W_1 is holomorphic in $\mathbb{C} \setminus S$. As this determination plays an important role in our discussion, it is worth reviewing in detail how this can be achieved. The standard determination of the square root gives a holomorphic function $x \mapsto \sqrt{x}$ on $\mathbb{C} \setminus \mathbb{R}_{\le 0}$ such that $\sqrt{\mathbb{R}_{>0}} = \mathbb{R}_{>0}$ and $(\sqrt{x})^2 = x$. We decompose $V'(x)^2 - 4P(x) = M(x)^2\sigma(x)$, where σ is a monic polynomial with simple real roots and M is a real polynomial with positive top coefficient. We write further

$$\sigma(x) = \prod_{h=0}^{g}(x - a_h)(x - b_h),$$

with:

$$a_0 < b_0 < a_1 < b_1 < \cdots < a_g < b_g,$$

The locus $\sigma^{-1}(\mathbb{C} \setminus \mathbb{R}_{\le 0})$ is a union of $g + 2$ connected components, labeled from left to right: C_0 contains a_0 in its closure, C_h contains b_h and a_{h+1} in its closure for $h \in [g - 1]$, and C_{g+1} contains b_g in its closure. For $x \in C_h$ with $h \in [0, g + 1]$ we set $s(x) = (-1)^{g+1-h}\sqrt{\sigma(x)}$. This definition makes $s(x)$ a continuous (thus holomorphic) function of $x \in \mathbb{C} \setminus S$. It is discontinuous on (a_h, b_h) because by crossing this segment we stay in the same component, so we keep the same global sign coming from the component we are in while the standard determination of the squareroot does get a sign change since

$\sigma(x)$ crosses $\mathbb{R}_{<0}$. Then, $M(x)s(x)$ is a holomorphic function of $x \in \mathbb{C} \setminus S$, and $M(x)s(x) \sim tx^{d-1}$ for some $d \geq 2$ and $t > 0$. The constraint $W_1(x) \sim \frac{1}{x}$ as $x \to \infty$ leads to the formula:

$$W_1(x) = \frac{V'(x) - M(x)s(x)}{2}. \tag{5.7}$$

The fact that W_1 is the Stieltjes transform of the equilibrium measure puts some constraints on the polynomial M.

Lemma 5.2.3. *The support of μ_{eq} is $S = \bigsqcup_{h=0}^{g} S_h$ with $S_h = [a_h, b_h]$ and we have*

$$\frac{\mathrm{d}\mu_{eq}}{\mathrm{d}x} = \frac{M(x)Im(s(x+\mathrm{i}0))}{2\pi}\mathbb{1}_S(x).$$

For each $h \in [g]$, the number of zeros (with multiplicity) of M in $[b_{h-1}, a_h]$ is odd. For each $h \in [0, g]$, the zeros of M in (a_h, b_h) have even multiplicity (if there is any).

Proof. By construction, for any $h \in [0, g]$ and $x \in (a_h, b_h)$ we have $s(x + \mathrm{i}0) \in (-1)^{g-h}\mathrm{i}\mathbb{R}_{>0}$ and for any $h \in [0, g+1]$ and $x \in (b_{h-1}, a_h)$ we have $s(x) \in (-1)^{g+1-h}\mathbb{R}_{>0}$, with the conventions $b_{-1} = -\infty$ and $a_{g+1} = +\infty$. By definition of the Stieltjes transform, the function $W_1(x)$ has a discontinuity in the interior of the support of μ_{eq}. It is identified as the *real* locus where the polynomial $V'(x)^2 - 4P(x)$ takes nonpositive values, and thus coincides with $S = \bigsqcup_{h=0}^{g}[a_h, b_h]$. The density of the equilibrium measure is reconstructed from the jump:

$$\begin{aligned} \forall x \in \mathbb{R}, \qquad \frac{\mathrm{d}\mu_{\text{eq}}}{\mathrm{d}x} &= \frac{W_1(x - \mathrm{i}0) - W_1(x + \mathrm{i}0)}{2\mathrm{i}\pi} \\ &= \frac{M(x)s(x+\mathrm{i}0)}{2\mathrm{i}\pi}\mathbb{1}_S(x) \\ &= \frac{M(x)\mathrm{Im}(s(x+\mathrm{i}0))}{2\pi}\mathbb{1}_S(x). \end{aligned}$$

Since μ_{eq} is a positive measure and since $\mathrm{Im}(s(x + \mathrm{i}0))$ has a constant sign in each (a_h, b_h), M should have constant sign in S_h and thus have zeros of even multiplicity there (if there is any). Likewise, since the sign of $\mathrm{Im}(s(x + \mathrm{i}0))$ changes between two consecutive segments in the support, M should have at least a sign change in the closure of the interval between these two segments, hence an odd number of zeros. ☐

Lemma 5.2.3 allows us to give an expression for the density ρ of the equilibrium measure. Indeed,

$$\begin{aligned} \rho(x) &= \frac{W_1(x - i0) - W_1(x + i0)}{2i\pi} = M(x)\frac{s(x+i0) - s(x-i0)}{4\pi}\mathbb{1}_S(x) \\ &= \frac{\sqrt{-M(x)\sigma(x)}}{2\pi}\mathbb{1}_S(x). \end{aligned} \tag{5.8}$$

Definition 5.2.4. *The effective potential is defined for $x \in A$ by $U(x) := V(x) - 2\int_A \ln|x - \xi|\mathrm{d}\mu_{eq}(\xi)$. It satisfies*

$$\begin{aligned} \forall x \in A \setminus S \qquad & U'(x) = V'(x) - 2W_1(x) = M(x)s(x), \\ \forall x \in \mathring{S} \qquad & U'(x) = V'(x) - W_1(x + \mathrm{i}0) - W_1(x - \mathrm{i}0) = 0. \end{aligned} \tag{5.9}$$

Remark 5.2.5. For the equilibrium measure in the unconstrained case, Theorem 5.2.1 says there exists a constant c such that $U(x) = c$ for all $x \in S$. In view of (5.9) the latter property is equivalent to

$$\forall h \in [g] \qquad \int_{b_{h-1}}^{a_h} M(x)s(x)\mathrm{d}x = 0. \tag{5.10}$$

5.2.4 Determinantal and pfaffian formulae

Expectation values of ratios of characteristic polynomials, also called kernels, are quantities of interest in random matrix theory. Let us introduce the notation $\langle\cdot\rangle_N^V$ for the expectation value with respect to $\mathbb{P}_N^V$ (the value of β will be specified in each case), and $\Lambda = \mathrm{diag}(\boldsymbol{\lambda})$. Given $c_1,\ldots,c_m \in \mathbb{Z}$, and $x_1,\ldots,x_m \in \mathbb{C}$ with the condition $x_j \notin A$ if $c_j < 0$, the m-point kernel is defined as

$$\left\langle \prod_{j=1}^{m} \det(x_j-\Lambda)^{c_j} \right\rangle_N^V = \left\langle \prod_{j=1}^{m}\prod_{i=1}^{N} (x_j-\lambda_i)^{c_j} \right\rangle_N^V .$$

In [BS06], Borodin and Strahov derive formulae to compute the kernels. In what follows, we will always consider the "balanced" case, that is, when there are as many characteristic polynomials in the numerator as in the denominator. Given two tuples of complex numbers $\boldsymbol{x} = (x_1,\ldots,x_{m_1})$ and $\tilde{\boldsymbol{x}} = (\tilde{x}_1,\ldots,\tilde{x}_{m_2})$, we write

$$\Delta(\boldsymbol{x},\tilde{\boldsymbol{x}}) = \prod_{i=1}^{m_1}\prod_{j=1}^{m_2}(x_i-\tilde{x}_j).$$

The determinantal case: $\beta = 2$

In that case, we have the following formulae.

Theorem 5.2.6. *[BS06, Theorem 4.1.1] Let N, m_1, m_2 be positive integers, and sets of complex numbers*

$$\begin{aligned} \boldsymbol{x} &= \{x_1,\ldots,x_{m_1}\} & \boldsymbol{x}' &= \{x'_1,\ldots,x'_{m_1}\} \\ \tilde{\boldsymbol{x}} &= \{\tilde{x}_1,\ldots,\tilde{x}_{m_2}\} & \tilde{\boldsymbol{x}}' &= \{\tilde{x}'_1,\ldots,\tilde{x}'_{m_2}\}, \end{aligned}$$

such that

$$\boldsymbol{x}\cap\boldsymbol{x}' = \emptyset, \qquad \tilde{\boldsymbol{x}}\cap\tilde{\boldsymbol{x}}' = \emptyset, \qquad \boldsymbol{x}'\cap A = \emptyset, \qquad \tilde{\boldsymbol{x}}'\cap A = \emptyset.$$

We have:

$$\left\langle \prod_{j=1}^{m_1} \frac{\det(x_j-\Lambda)}{\det\left(x'_j-\Lambda\right)} \prod_{j=1}^{m_2} \frac{\det(\tilde{x}_j-\Lambda)}{\det\left(\tilde{x}'_j-\Lambda\right)} \right\rangle_N^V = (-1)^{\frac{1}{2}((m_1+m_2)^2+m_2-m_1)} \frac{\Delta(\boldsymbol{x},\boldsymbol{x}')}{\Delta(\boldsymbol{x})\Delta(\boldsymbol{x}')} \frac{\Delta(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{x}}')}{\Delta(\tilde{\boldsymbol{x}})\Delta(\tilde{\boldsymbol{x}}')} \times \det\left(\mathcal{M}^{(2)}(\boldsymbol{x},\boldsymbol{x}';\tilde{\boldsymbol{x}},\tilde{\boldsymbol{x}}')\right), \tag{5.11}$$

in terms of the block matrix of size (m_1+m_2):

$$\mathcal{M}^{(2)}(\boldsymbol{x},\boldsymbol{x}';\tilde{\boldsymbol{x}},\tilde{\boldsymbol{x}}') = \begin{pmatrix} \mathcal{M}^{(2)}_{++}(x_i,\tilde{x}_j) & \mathcal{M}^{(2)}_{+-}(x_i,x'_j) \\ \mathcal{M}^{(2)}_{-+}(\tilde{x}'_i,\tilde{x}_j) & \mathcal{M}^{(2)}_{--}(\tilde{x}'_i,x'_j) \end{pmatrix},$$

where i is a row index, j a column index, and the entries are:

$$\begin{aligned} \mathcal{M}^{(2)}_{++}(x,\tilde{x}) &= N\frac{Z^{\frac{N}{N-1}V}_{N-1}}{Z^V_N} \left\langle \det(x-\Lambda)\det(\tilde{x}-\Lambda)\right\rangle^{\frac{N}{N-1}V}_{N-1}, \\ \mathcal{M}^{(2)}_{+-}(x,x') &= \frac{1}{x-x'}\left\langle \frac{\det(x-\Lambda)}{\det(x'-\Lambda)}\right\rangle^V_N, \\ \mathcal{M}^{(2)}_{-+}(\tilde{x}',\tilde{x}) &= \frac{1}{\tilde{x}'-\tilde{x}}\left\langle \frac{\det(\tilde{x}-\Lambda)}{\det(\tilde{x}'-\Lambda)}\right\rangle^V_N, \\ \mathcal{M}^{(2)}_{--}(\tilde{x}',x') &= \frac{1}{N+1}\frac{Z^{\frac{N}{N+1}V}_{N+1}}{Z^V_N} \left\langle \frac{1}{\det(\tilde{x}'-\Lambda)\det(x'-\Lambda)}\right\rangle^{\frac{N}{N+1}V}_{N+1}. \end{aligned}$$

Orthogonal ensembles: $\beta = 1$

Recall that for an antisymmetric matrix A of size $2m$, the pfaffian is defined as

$$\mathrm{Pf}(A) = \frac{1}{m!2^m} \sum_{\sigma \in \mathfrak{S}_{2m}} \mathrm{sgn}(\sigma) \prod_{i=1}^{m} A_{\sigma(2i-1),\sigma(2i)}.$$

Theorem 5.2.7. *[BS06, Theorem 1.2.1] Let N, m be positive integers and set of complex numbers*

$$\boldsymbol{x} = \{x_1, \ldots, x_m\}, \qquad \boldsymbol{x}' = \{x'_1, \ldots, x'_m\},$$

such that $\boldsymbol{x}' \cap A = \emptyset$. We have:

$$\left\langle \prod_{j=1}^{m} \frac{\det(x_j - \Lambda)}{\det\left(x'_j - \Lambda\right)} \right\rangle_{2N}^{V} = \frac{\Delta(\boldsymbol{x}, \boldsymbol{x}')}{\Delta(\boldsymbol{x})\Delta(\boldsymbol{x}')} \mathrm{Pf}\left(\mathcal{M}^{(1)}(\boldsymbol{x}, \boldsymbol{x}')\right), \tag{5.12}$$

in terms of the antisymmetric matrix of size $2m$:

$$\mathcal{M}^{(1)}(\boldsymbol{x}, \boldsymbol{x}') = \begin{pmatrix} \mathcal{M}^{(1)}_{++}(x_i, x_j) & \mathcal{M}^{(1)}_{+-}(x_i, x'_j) \\ \mathcal{M}^{(1)}_{-+}(x'_i, x_j) & \mathcal{M}^{(1)}_{--}(x'_i, x'_j) \end{pmatrix}, \tag{5.13}$$

with entries

$$\begin{aligned} \mathcal{M}^{(1)}_{++}(x, \tilde{x}) &= (2N-1)2N(x - \tilde{x}) \frac{Z^{\frac{2N}{2N-2}V}_{2N-2}}{Z^{V}_{2N}} \left\langle \det(x - \Lambda) \det(\tilde{x} - \Lambda) \right\rangle^{\frac{2N}{2N-2}V}_{2N-2}, \\ \mathcal{M}^{(1)}_{+-}(x, x') &= \frac{1}{x - x'} \left\langle \frac{\det(x - \Lambda)}{\det(x' - \Lambda)} \right\rangle^{V}_{2N} = -\mathcal{M}^{(1)}_{-+}(x', x), \\ \mathcal{M}^{(1)}_{--}(x', \tilde{x}') &= \frac{x' - \tilde{x}'}{(2N+1)(2N+2)} \frac{Z^{\frac{2N}{2N+2}V}_{2N+2}}{Z^{V}_{2N}} \left\langle \frac{1}{\det(x' - \Lambda) \det(\tilde{x}' - \Lambda)} \right\rangle^{\frac{2N}{2N+2}V}_{2N+2}. \end{aligned} \tag{5.14}$$

Symplectic ensembles: $\beta = 4$

The case $\beta = 4$ is very similar to the case $\beta = 1$.

Theorem 5.2.8. *[BS06, Theorem 1.2.1] Let N, m be positive integers and two sets of complex numbers*

$$\boldsymbol{x} = \{x_1, \ldots, x_m\}, \qquad \boldsymbol{x}' = \{x'_1, \ldots, x'_m\},$$

such that $\boldsymbol{x}' \cap A = \emptyset$. We have

$$\left\langle \prod_{j=1}^{m} \frac{\det(x_j - \Lambda)^2}{\det\left(x'_j - \Lambda\right)^2} \right\rangle_{N}^{V} = \frac{\Delta(\boldsymbol{x}, \boldsymbol{x}')}{\Delta(\boldsymbol{x})\Delta(\boldsymbol{x}')} \mathrm{Pf}\left(\mathcal{M}^{(4)}(\boldsymbol{x}, \boldsymbol{x}')\right), \tag{5.15}$$

with the same block structure as (5.13) but entries

$$\begin{aligned} \mathcal{M}^{(4)}_{++}(x, \tilde{x}) &= N \frac{Z^{\frac{N}{N-1}V}_{N-1}}{Z^{V}_{N}} (x - \tilde{x}) \left\langle \det(x - \Lambda)^2 \det(\tilde{x} - \Lambda)^2 \right\rangle^{\frac{N}{N-1}V}_{N-1}, \\ \mathcal{M}^{(4)}_{+-}(x, x') &= \frac{1}{x - x'} \left\langle \frac{\det(x - \Lambda)^2}{\det(x' - \Lambda)^2} \right\rangle^{V}_{N} = -\mathcal{M}^{(4)}_{-+}(x', x), \\ \mathcal{M}^{(4)}_{--}(x', \tilde{x}') &= \frac{1}{N+1} \frac{Z^{\frac{N}{N+1}V}_{N+1}}{Z^{V}_{N}} (x' - \tilde{x}') \left\langle \frac{1}{\det(x' - \Lambda)^2 \det(\tilde{x}' - \Lambda)^2} \right\rangle^{\frac{N}{N+1}V}_{N}. \end{aligned} \tag{5.16}$$

5.3 Geometry of the spectral curves

This section collects the information on theta functions and geometry of the spectral curve that will be needed later to present the large N asymptotics in the β-ensembles. We only give the details necessary to understand the formulae of Section 5.4 and 5.5 in a self-contained way. We refer to the many textbooks address in details theta functions and the geometry of Riemann surfaces, for instance [Fay73; Mum83; FK92; Ber06].

5.3.1 Theta functions

Let us recall the definition and properties of the theta function.

Definition 5.3.1. *Let $\boldsymbol{\tau}$ be a complex $g \times g$ symmetric matrix such that $\mathrm{Im}\,\boldsymbol{\tau}$ is positive definite. The theta function with characteristics $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{R}^g$ is the function defined by*

$$\forall \boldsymbol{z} \in \mathbb{C}^g \qquad \vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}(\boldsymbol{z}|\boldsymbol{\tau}) = \sum_{\boldsymbol{n} \in \mathbf{Z}^g} \exp\big(\mathrm{i}\pi(\boldsymbol{n}+\boldsymbol{\mu}) \cdot \boldsymbol{\tau}(\boldsymbol{n}+\boldsymbol{\mu}) + 2\mathrm{i}\pi(\boldsymbol{n}+\boldsymbol{\mu}) \cdot (\boldsymbol{z}+\boldsymbol{\nu})\big).$$

We set $\theta := \vartheta_{\mathbf{0},\mathbf{0}}$.

The condition $\mathrm{Im}\,\boldsymbol{\tau} > 0$ ensures that the function is well defined. Let us define the period lattice associated to $\boldsymbol{\tau}$ as $\mathbb{L} = \mathbf{Z}^g \oplus \boldsymbol{\tau}(\mathbf{Z}^g)$. The theta function is quasi-periodic: for $\boldsymbol{m}, \boldsymbol{n} \in \mathbf{Z}^g$ we have $\boldsymbol{m} + \boldsymbol{\tau}(\boldsymbol{n}) \in \mathbb{L}$, and for any $\boldsymbol{z} \in \mathbb{C}^g$

$$\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(\boldsymbol{z}+\boldsymbol{m}+\boldsymbol{\tau}(\boldsymbol{n})\big|\boldsymbol{\tau}\big) = e^{2\mathrm{i}\pi \boldsymbol{m}\cdot\boldsymbol{\mu} - \mathrm{i}\pi\boldsymbol{n}\cdot(\boldsymbol{\tau}(\boldsymbol{n})+2\boldsymbol{z}+2\boldsymbol{\nu})}\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}(\boldsymbol{z}|\boldsymbol{\tau}).$$

Definition 5.3.2. *An odd half-integer characteristic is $\boldsymbol{c} = \frac{1}{2}\boldsymbol{e} + \frac{1}{2}\boldsymbol{\tau}(\boldsymbol{e}')$, with $\boldsymbol{e}, \boldsymbol{e}' \in \mathbf{Z}^g$ such that $\boldsymbol{e} \cdot \boldsymbol{e}' \in 2\mathbb{Z}+1$.*

By direct computation, if $\boldsymbol{c}$ is a odd half-integer characteristic, then $\theta(\boldsymbol{c}|\boldsymbol{\tau}) = 0$.

5.3.2 Geometry of Riemann surfaces

Basis of cycles and forms

Let $\hat{C}$ be a compact Riemann surface of genus g. The first homology group $H_1(\hat{C};\mathbb{Z})$ has a basis $(\mathcal{A}_h, \mathcal{B}_h)_{h=1}^g$ which can be chosen to have the following properties under the intersection pairing

$$\forall h,k \in [g] \qquad \mathcal{A}_h \cap \mathcal{A}_k = 0, \qquad \mathcal{B}_h \cap \mathcal{B}_k = 0, \qquad \mathcal{A}_h \cap \mathcal{B}_k = \delta_{h,k}.$$

Such a basis is called a symplectic basis of homology, and $\hat{C}$ equipped with such a basis is called a marked Riemann surface. We can for instance choose a point p_0 and simple closed curves on $\hat{C}$ representing the $2g$ classes $(\mathcal{A}_h, \mathcal{B}_h)_{h=1}^g$ such that all the curves intersect each other at p_0 only. For spectral curves of β-ensembles, we will later work with another set of representatives (Section 5.3.3). We keep the same notation for homology classes and their representatives. The surface $\hat{C}^0 = \hat{C} \setminus \bigcup_{h=1}^g(\mathcal{A}_h \cup \mathcal{B}_h)$ is then simply-connected. The $\mathcal{A}$-cycles $(\mathcal{A}_h)_{h=1}^g$ determine a dual basis of holomorphic 1-forms $(\mathrm{d}u_h)_{h=1}^g$, such that

$$\forall h,k \in [g] \qquad \oint_{\mathcal{A}_h} \mathrm{d}u_k = \delta_{h,k}.$$

The matrix of periods $\boldsymbol{\tau}$ is then defined by

$$\forall h,k \in [g] \qquad \tau_{h,k} = \oint_{\mathcal{B}_h} \mathrm{d}u_k. \tag{5.17}$$

It is symmetric and $\mathrm{Im}(\boldsymbol{\tau})$ is definite positive, in particular we can consider the theta function with matrix $\frac{\beta}{2}\boldsymbol{\tau}$ for any $\beta > 0$. The theta function with matrix equal to (5.17) is called the Riemann theta function.

Abel map

With the 1-forms $(\mathrm{d}u_h)_{h=1}^{g}$ defined in Section 5.3.2 we can introduce the Abel map.

Definition 5.3.3. *Choose a base point p_0 in $\hat{C}$. The Abel map $\boldsymbol{u}\colon \hat{C}^0 \to \mathbb{C}^g$ is defined by*

$$u_i(z) = \int_{p_0}^{z} \mathrm{d}u_i,$$

where the path of integration is in $\hat{C}^0$.

The definition of the Abel map depends on a choice of base point p_0. However, we will often consider differences $\boldsymbol{u}(z) - \boldsymbol{u}(w)$ of Abel maps, which are independent of p_0. Depending on the context, we may also consider the Abel map as a map $\boldsymbol{u}\colon \widetilde{C} \to \mathbb{C}^g$ defined on the universal cover $\widetilde{C}$ of $\hat{C}$ based at p_0. We say that $\boldsymbol{c}$ is nonsingular if $\theta\big(\boldsymbol{c} + \boldsymbol{u}(z) - \boldsymbol{u}(w)\big|\boldsymbol{\tau}\big)$ is not identically 0 when $z, w \in \widetilde{C}$. Non singular odd half-integer characteristics exist, and in what follows we fix one.

Prime form

We introduce the holomorphic 1-form

$$\omega_{\boldsymbol{c}} = \sum_{h=1}^{g} \partial_{z_h} \theta(\boldsymbol{z}|\boldsymbol{\tau})\big|_{\boldsymbol{z}=\boldsymbol{c}} \,\mathrm{d}u_h.$$

The prime form is:

$$E(z_1, z_2) = \frac{\theta(\boldsymbol{c} + \boldsymbol{u}(z_1) - \boldsymbol{u}(z_2)|\boldsymbol{\tau})}{\sqrt{\omega_{\boldsymbol{c}}(z_1)}\sqrt{\omega_{\boldsymbol{c}}(z_2)}}. \tag{5.18}$$

It is defined as a holomorphic bispinor on $\widetilde{C} \times \widetilde{C}$, i.e. a $(-\frac{1}{2}) \otimes (-\frac{1}{2})$ form. It has zeros only at $z_1 = z_2$, and in local coordinates ζ, we have

$$E(z_1, z_2) \underset{z_1 \to z_2}{\sim} \frac{\zeta(z_1) - \zeta(z_2)}{\sqrt{\mathrm{d}\zeta(z_1)\mathrm{d}\zeta(z_2)}}.$$

The prime form on the Riemann sphere $\hat{\mathbb{C}}$ reads

$$E_0(x_1, x_2) = \frac{x_1 - x_2}{\sqrt{\mathrm{d}x_1 \mathrm{d}x_2}}.$$

Given a meromorphic function $X: \hat{C} \to \hat{\mathbb{C}}$, we can define the relative prime form:

$$\tilde{E}(z_1, z_2) = \frac{E(z_1, z_2)}{E_0(X(z_1), X(z_2))}. \tag{5.19}$$

We observe that $\tilde{E}(z_1, z_2)$ is a function on $\widetilde{C} \times \widetilde{C}$, such that

$$\lim_{z_2 \to z_1} \tilde{E}(z_1, z_2) = 1. \tag{5.20}$$

Fundamental bidifferential

Definition 5.3.4. *The fundamental bidifferential $B(z, w)$ is the unique bidifferential (i.e. a $1 \otimes 1$ form on $\hat{C} \times \hat{C}$) such that*

1. *Symmetry: $B(z, w) = B(w, z)$;*
2. *Normalisation: $\forall h \in [g] \quad \oint_{\mathcal{A}_h} B(\cdot, w) = 0$;*

3. *Singularities: $B(z,w)$ is meromorphic with only a double pole at $z = w$, and if ζ is a local coordinate, we have*

$$B(z,w) \underset{z\to w}{=} \left(\frac{1}{(\zeta(z)-\zeta(w))^2} + S_{B,\zeta}(w) + \mathcal{O}(\zeta(z)-\zeta(w))\right) \mathrm{d}\zeta(z)\mathrm{d}\zeta(w).$$

for some function $S_{B,\zeta}$ locally defined on $\hat{C}$.

The fundamental bidifferential can be expressed as

$$B(z,w) = \mathrm{d}_z\mathrm{d}_w \ln\theta(\boldsymbol{c} + \boldsymbol{u}(z) - \boldsymbol{u}(w)).$$

and this expression is independent on the choice of a nonsingular odd half-integer characteristics $\boldsymbol{c}$. Given $p, q \in \hat{C}$ and a choice of path $\gamma_{p,q}$ from p to q, we define the meromorphic form

$$\mathrm{d}S_{p,q}(z) = \int_{\gamma_{p,q}} B(z,\cdot),$$

It has two poles of order 1 in p and q, with respective residue -1 and $+1$. Equivalently we can consider that it is specified by the choice of two points p and q in the universal cover $\widetilde{C}$. The prime form appears in the following computation.

Lemma 5.3.5. *For $i = 1, 2$, let $z_i, \tilde{z}_i \in \widetilde{C}$ and γ_i a path from $\tilde{z}_i$ to z_i. Then:*

$$\int_{\gamma_1}\int_{\gamma_2} B = \int_{\gamma_1} \mathrm{d}S_{\tilde{z}_2,z_2} = \ln\left(\frac{E(z_1,z_2)E(\tilde{z}_1,\tilde{z}_2)}{E(z_1,\tilde{z}_2)E(\tilde{z}_1,z_2)}\right).$$

Decomposition of meromorphic forms

One distinguishes between three kinds of meromorphic forms:

- holomorphic 1-forms (first kind);
- meromorphic 1-forms with vanishing residues (second kind);
- meromorphic 1-form with non-vanishing residues (third kind).

The space of first kind differentials has for basis $(\mathrm{d}u_h)_{h=1}^{g}$, which are dual to the $\mathcal{A}$-cycles.

Assume that a choice of local coordinate ζ_p near each point $p \in \hat{C}$ has been made. A basis of the space of second kind differentials is then given by

$$\mathrm{d}B_{p,k}(z) = \operatorname*{Res}_{z'=p} \zeta_p(z')^{-k} B(z',z).$$

for $p \in \hat{C}$ and $k \in \mathbb{Z}_{>0}$. Given the properties of the fundamental bidifferential, its only pole is at p with order $(k+1)$, where it behaves like:

$$\mathrm{d}B_{p,k}(z) \underset{z\to p}{=} -\mathrm{d}\Big(\zeta_p(z)^{-k}\Big) + \mathcal{O}(\mathrm{d}\zeta_p(z)).$$

Besides, we have $\oint_{\mathcal{A}_h} \mathrm{d}B_{p,k} = 0$ for any $h \in [g]$.

Assume that for each $p \in \hat{C}^0$, a choice of path $\gamma_{p_0,p}$ from p_0 to p in $\hat{C}^0$ has been made. An example of third kind differential is

$$\mathrm{d}S_{p_0,p}(z) = \int_{\gamma_{p_0,p}} B(z,\cdot), \tag{5.21}$$

It has two poles of order 1 in p_0 and p, respectively of residue -1 and $+1$, and has zero $\mathcal{A}$-periods.

Every meromorphic 1-form ϕ can be decomposed uniquely as a sum of first kind, second kind and third kind differentials[2]:

$$\begin{aligned}\phi(z) = &\Big(\sum_{h=1}^{g} \oint_{\mathcal{A}_h} \phi\Big)\mathrm{d}u_h(z) \\ &+ \sum_{\substack{p \text{ simple pole} \\ \text{of } \phi}} \big(\operatorname*{Res}_p \phi\big)\mathrm{d}S_{p_0,p}(z) \\ &+ \sum_{k\geq 1} \sum_{\substack{p \text{ pole of} \\ \text{order } (k+1) \text{ of } \phi}} \big(\operatorname*{Res}_p \zeta_p^k \phi\big)\,\frac{\mathrm{d}B_{p,k}(z)}{k}.\end{aligned}$$

5.3.3 The spectral curve

We elaborate on Section 5.2.3 and construct the spectral curve associated to the equilibrium measure of β-ensembles, for the moment indifferently in the unconstrained case or the fixed filling fraction case. This prepares us for Section 5.4 where the asymptotics in the β-ensembles is described solely in terms of the geometry of this spectral curve.

Construction of the marked Riemann surface

The equation (5.5) satisfied by the Stieltjes transform of the equilibrium measure has two solutions:

$$\begin{aligned} F_+(x) &= \frac{V'(x)}{2} + y(x) = W_1(x), \\ F_-(x) &= \frac{V'(x)}{2} - y(x) = V'(x) - W_1(x). \end{aligned} \tag{5.22}$$

where $y(x) = -\frac{1}{2}M(x)s(x)$ and $y(x)^2 = \frac{1}{4}V'(x)^2 - P(x)$. After the birational transformation $(x,y) \mapsto (x, s = -\frac{2y}{M(x)})$ we have the equation of an hyperelliptic curve

$$s^2 = \sigma(x) = \prod_{h=0}^{g}(x-a_h)(x-b_h)$$

where the Weierstraß points $a_0, b_0, \ldots, a_g, b_g$ are real. This curve is constructed from two sheets homeomorphic to $\mathbb{C}\setminus S$, which are glued together along S. The two sheets are embedded into the curve by

$$\iota_\pm : \begin{array}{ccc} \mathbb{C}\setminus S & \longrightarrow & C \subset \mathbb{C}\times\mathbb{C} \\ x & \longmapsto & (x, \pm s(x)) \end{array}.$$

We denote the sheets by $C_\pm = \iota_\pm(\mathbb{C}\setminus S)$, and $\hat{C}_\pm$ are the sheets including their point at infinity. Adding these points at infinity $\infty_\pm$ to C, we get a compact Riemann surface $\hat{C}$. We define the projection map as the meromorphic function

$$X : \begin{array}{ccc} \hat{C} & \longrightarrow & \hat{\mathbb{C}} \\ (x,s) & \longmapsto & x \end{array}.$$

This function has simple poles at $\infty_\pm$ and it defines a degree 2 branched covering of the Riemann sphere $\hat{\mathbb{C}}$, whose branch points are the zeros of σ. It allows us to define local coordinates ζ_p around any point $p \in \hat{C}$, which is used in Section 5.3.2 to define a basis of meromorphic differentials:

[2] Given ϕ, we can always perturb the representatives of $\mathcal{A}$ and $\mathcal{B}$-cycles so that all poles are contained in $\hat{C}^0$ and we can use (5.21).

- If p is a ramification point, we take $\zeta_p(z) = \sqrt{X(z) - X(p)}$ for some choice of sign for the squareroot.
- If $p = \infty_\pm$, then $\zeta_{\infty_\pm} = X(z)^{-1}$.
- In all the other cases, $\zeta_p(z) = X(z) - X(p)$.

We shall take $p_0 = \infty_+$ as reference point for the definition of the Abel map (Section 5.3.2). For later use we analyse the prime form over $\hat{\mathbb{C}}$ near $\infty_\pm$.

Lemma 5.3.6. *We have:*

$$E_0(z, \tilde{z})\sqrt{\mathrm{d}\zeta_{\infty_\pm}(\tilde{z})}\Big|_{\tilde{z}=\infty_\pm} = \frac{-1}{\sqrt{-\mathrm{d}X(z)}}.$$

Proof. Since $\mathrm{d}X(\tilde{z}) = -X(\tilde{z})^2\mathrm{d}\zeta_{\infty_\pm}(\tilde{z})$, we have

$$\lim_{\tilde{z}\to\infty_\pm} E_0(z, \tilde{z})\sqrt{\mathrm{d}\zeta_{\infty_\pm}(\tilde{z})} = \lim_{\tilde{z}\to\infty_\pm} (X(z) - X(\tilde{z}))\frac{\sqrt{-X(\tilde{z})^{-2}\mathrm{d}X(\tilde{z})}}{\sqrt{\mathrm{d}X(z)\mathrm{d}X(\tilde{z})}} = \frac{-1}{\sqrt{-\mathrm{d}X(z)}}.$$

□

We choose representatives for a symplectic basis of homology on $\hat{C}$ like in Figure 5.1. Namely, we take $\mathcal{A}_h$ representing a counterclockwise loop in C_+ going around the cut S_h, for $h \in [g]$. For convenience we fix a representative $\mathcal{A}_0$ of a counterclockwise loop surrounding S_0 in C_+, whose homology class is $-(\mathcal{A}_1 + \cdots + \mathcal{A}_g)$. We take $\mathcal{B}_h$ representing a loop in $\hat{C}$ travelling from S_0 to S_h in C_+ and in the opposite direction in C_-.

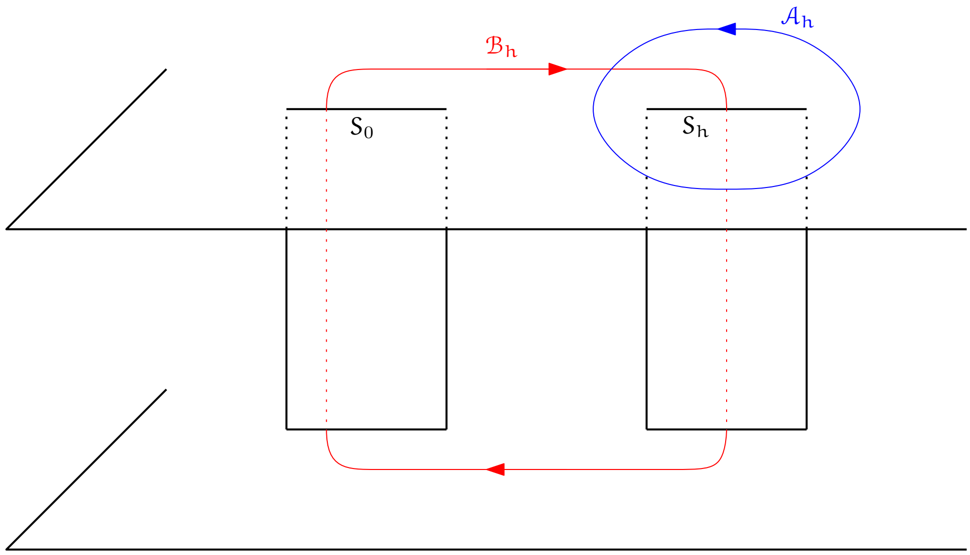


Figure 5.1: Two cycles $\mathcal{A}_h$ and $\mathcal{B}_h$.

The 1-form ϕ and the Stieltjes transform

The branched covering $X : \hat{C} \to \hat{\mathbb{C}}$ alone does not determine the equilibrium measure: we also need to specify the meromorphic function $Y : (x, s) \mapsto -\frac{1}{2}M(x)s$ on $\hat{C}$. It is such that $Y(x) = \mp\frac{1}{2}M(x)s(x)$ for $x \in \hat{C}_\pm$. The advantage to work with the Riemann surface $\hat{C}$ is that the function W_1, originally defined on $\mathbb{C} \setminus S$ (Section 5.2.3) can be analytically continued to a meromorphic function on the whole $\hat{C}$. Indeed, the meromorphic function $\mathcal{W}_1(z) = \frac{V'(X(z))}{2} + Y(z)$ coincides with $W_1(X(z))$ for $z \in \hat{C}_+$. We can also see this by noticing that $F_\pm$ in (5.22) are two solutions of the Riemann–Hilbert problem

$$\forall x \in \mathring{S} \qquad F(x + \mathrm{i}0) + F(x - \mathrm{i}0) = V'(x). \tag{5.23}$$

Definition 5.3.7. *We equip $\hat{C}$ with the meromorphic 1-form $\phi(z) = \mathcal{W}_1(z)\mathrm{d}X(z)$.*

The previous discussion shows that ϕ has a simple pole at ∞_+ with residue -1, and a higher order pole at ∞_- with:

$$\phi(z) \underset{z\to\infty_-}{=} \mathrm{d}V(X(z)) - \frac{\mathrm{d}X(z)}{X(z)} + \mathcal{O}\bigg(\frac{\mathrm{d}X(z)}{X(z)^2}\bigg),$$

where we recall that $\zeta_{\infty_\pm} = 1/X$ is a local coordinate near $\infty_\pm$. As X has two simple poles at $\infty_\pm$, the form $\mathrm{d}X$ thus has double poles at these points with

$$\mathrm{d}X = -\zeta_{\infty_\pm}^2 \mathrm{d}\zeta_{\infty_\pm}.$$

The 1-form ϕ therefore decomposes as in Section 5.3.2:

$$\phi = \sum_{h=1}^{g} 2\mathrm{i}\pi\, \epsilon_h^* \mathrm{d}u_h + \mathrm{d}S_{\infty_+,\infty_-} - \sum_{k=1}^{d} \frac{t_k}{k} \mathrm{d}B_{\infty_-,k}. \tag{5.24}$$

where the potential is $V(x) = \sum_{k=1}^{d} \frac{t_k x^k}{k}$. The path from ∞_+ to ∞_- used in the definition of the 1-form $\mathrm{d}S_{\infty_+,\infty_-}$ is chosen so that it does not intersect $(\mathcal{A}_h, \mathcal{B}_h)_{h=1}^{g}$. For instance, one can take it to be $\iota_+(a_0 + \mathrm{i}\mathbb{R}_{\geq 0}) \cup \iota_-(a_0 - \mathrm{i}\mathbb{R}_{\leq 0})$.

Remark 5.3.8. For the equilibrium measure in the unconstrained case, the property noticed in Remark 5.2.5 can be equivalently rewritten as

$$\forall h \in [g] \qquad \oint_{\mathcal{B}_h} \phi = 0.$$

As the filling fractions are real, this implies that for any $\gamma \in H_1(\hat{C}, \mathbb{Z})$, we have $\mathrm{Re}\big(\oint_\gamma \phi\big) = 0$. Pairs $(\hat{C}, \phi)$ satisfying this property are called Boutroux curves, see [Ber11].

The fundamental bidifferential and the second correlator

Under the assumptions discussed in Section 5.4.1, [BG24] shows that the $N \to \infty$ limit of the second correlator in the model with fixed filling fractions $\boldsymbol{N}/N \to \boldsymbol{\epsilon}$ exists:

$$W_2(x_1, x_2) = \lim_{N\to\infty} \frac{\beta}{2}\bigg(\Big\langle \mathrm{Tr}\,\Big(\frac{1}{x_1 - \Lambda}\Big)\, \mathrm{Tr}\,\Big(\frac{1}{x_2 - \Lambda}\Big)\Big\rangle_{N,\boldsymbol{N}/N}^{V} - W_1(x_1)W_1(x_2)\bigg).$$

It can be shown to satisfy the Riemann–Hilbert problem

$$\forall (x_1, x_2) \in (\mathbb{C} \setminus S) \times \mathring{S}, \qquad W_2(x_1, x_2 + \mathrm{i}0) + W_2(x_1, x_2 - \mathrm{i}0) = -\frac{1}{(x_1 - x_2)^2}, \tag{5.25}$$

for $x \in C \setminus S$ and $y \in S$, see for instance [EKR18, Chapter 3] or [BEO15]. Besides, $W_2(x_1, x_2) = \mathcal{O}\big(1/x_i^2\big)$ as $x_i \to \infty$ since the total number of particles is deterministic, and

$$\forall h \in [g] \qquad \oint_{\mathcal{A}_h} W_2(x_1, x_2)\mathrm{d}x_1 = 0, \tag{5.26}$$

since the filling fraction of the segment A_h is fixed.

The Riemann–Hilbert problem (5.25) implies that we can define a meromorphic function $\mathcal{W}_2(z_1, z_2)$ on $\hat{C} \times \hat{C}$ such that $\mathcal{W}_2(z_1, z_2) = W_2(X(z_1), X(z_2))$ when $z_1, z_2 \in \hat{C}_+$. By examining the behavior of $\mathcal{W}_2$ at the poles and considering the $\mathcal{A}$-period conditions (5.26), one can identify it in terms of the fundamental bidifferential:

$$B(z_1, z_2) = \mathcal{W}_2(z_1, z_2)\mathrm{d}X(z_1)\mathrm{d}X(z_2) + \frac{\mathrm{d}X(z_1)\mathrm{d}X(z_2)}{(X(z_1) - X(z_2))^2}. \tag{5.27}$$

Definition 5.3.9. *The spectral curve of a β-ensemble is the marked compact Riemann surface $(\hat{C}, \mathcal{A}, \mathcal{B})$ equipped with the meromorphic functions X, Y and the bidifferential B.*

5.3.4 Characterisation of spectral curves of β-ensembles

In Section 5.3.3 we explained that the Riemann surface $\hat{C}$ underlying the spectral curve of a β-ensemble is hyperelliptic with real Weierstraß points. We now prove the converse, namely that all such Riemann surfaces can be realised (non uniquely) as the underlying Riemann surface of the spectral curve of an unconstrained β-ensemble.

Definition 5.3.10. *If G is a meromorphic function in a neighborhood of ∞ in $\hat{\mathbb{C}}$, we define its polynomial part $\mathcal{V}[G](x)$, which is the unique polynomial such that $G(x) = \mathcal{V}[G](x) + \mathcal{O}\big(\frac{1}{x}\big)$ as $x \to \infty$.*

Proposition 5.3.11. *For any $a_0 < b_0 < \cdots < a_g < b_g$, there exists a polynomial V of degree $(2g+2)$ with top coefficient $\frac{t_{2g+2}}{2g+2} > 0$ and there exist for each $h \in [0,g]$ a segment A_h which is a neighborhood of $[a_h, b_h]$ in $\mathbb{R}$, such that the unconstrained β-ensemble with potential V on $A = \bigsqcup_{h=0}^{g} A_h$ admits an equilibrium measure with support $S = \bigsqcup_{h=0}^{g}[a_h, b_h]$ and in (5.7) we have $M(x) = t_{2g+2}\prod_{h=1}^{g}(x - z_h)$ having roots outside A and such that $b_{h-1} < z_h < a_h$ for any $h \in [g]$.*

Proof. Take $2g+2$ real points $a_0 < b_0 < \cdots < a_g < b_g$ and introduce polynomials $\sigma(x) = \prod_{h=0}^{g}(x - a_h)(x - b_h)$. We have seen in Section 5.2.3 that there exists a unique holomorphic function $s(x)$ on $\mathbb{C} \setminus \bigsqcup_{h=0}^{g}[a_h, b_h]$ such that $s(x)^2 = \sigma(x)$ and $s(x) \sim x^{g+1}$ as $x \to \infty$ in the complex plane. Take $h \in [g]$ and introduce the continuous function

$$\forall \boldsymbol{\lambda} \in [0,1]^g \qquad J_h(\boldsymbol{\lambda}) = \int_{b_{h-1}}^{a_h} s(x) \prod_{k=1}^{g} \big(x - (\lambda_k b_{k-1} + (1-\lambda_k)a_k)\big)\mathrm{d}x.$$

By continuity, $s(x)$ has constant sign for $x \in (b_{h-1}, a_h)$. Besides, for $\lambda_h \in \{0,1\}$ we have $\mathrm{sgn}(J_h(\boldsymbol{\lambda})) = (-1)^{g-h+1-\lambda_h}$. The Poincaré–Miranda theorem [Mir41] then implies the existence of $\boldsymbol{\lambda}^* \in [0,1]^g$ such that $J_h(\boldsymbol{\lambda}^*)$ vanishes for any $h \in [g]$. Since $J_h(\boldsymbol{\lambda})$ does not vanish when $\lambda_h \in \{0,1\}$, this $\boldsymbol{\lambda}^*$ must be in $(0,1)^g$. We let $z_h = \lambda_h^* b_{h-1} + (1-\lambda_h^*)a_h$ for $h \in [g]$ and introduce the polynomial $M(x) = t_{2g+2}\prod_{h=1}^{g}(x - z_h)$, where the constant t_{2g+2} is chosen such that

$$\sum_{h=0}^{g} \int_{a_h}^{b_h} \frac{M(x)\mathrm{Im}(s(x+\mathrm{i}0))}{2\pi} = 1. \tag{5.28}$$

The sign discussion for s in Section 5.2.3 reveals that all terms in (5.28) are positive, thus $t_{2g+2} > 0$. Then, $V(x) = \int_0^x \mathcal{V}[M \cdot s](\xi)\mathrm{d}\xi$ is a polynomial of degree $2g+2$ with top coefficient $\frac{t_{2g+2}}{2g+2}$, and

$$\operatorname*{Res}_{x=\infty} \frac{s(x)M(x)}{2}\mathrm{d}x = -\frac{1}{2\mathrm{i}\pi}\sum_{h=0}^{g}\oint_{\mathcal{A}_h} \frac{s(x)M(x)}{2}\mathrm{d}x = \sum_{h=0}^{g}\int_{a_h}^{b_h} \frac{M(x)\mathrm{Im}(s(x+\mathrm{i}0))}{2\pi} = 1.$$

where $\mathcal{A}_h$ is a counterclockwise loop around $[a_h, b_h]$. Therefore,

$$\frac{V'(x)}{2} \underset{x\to\infty}{=} \frac{M(x)s(x)}{2} + \frac{1}{x} + \mathcal{O}\Big(\frac{1}{x^2}\Big).$$

This V defines the potential in a β-ensemble which we consider over the domain $A = \bigsqcup_{h=0}^{g} A_h$, where $A_h = [a_h', b_h']$ and $z_h < a_h' < a_h$ and $b_h < b_h' < z_{h+1}$ for any $h \in [0,g]$, with the conventions $z_0 = -\infty$ and $z_{g+1} = +\infty$. It remains to check that $W(x) := \frac{1}{2}(V'(x) - M(x)s(x))$ is the Stieltjes transform of the equilibrium measure of this (unconstrained) β-ensemble.

We define the measure μ with support $S = \bigsqcup_{h=0}^{g}[a_h, b_h]$ and density

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{W(x - \mathrm{i}0) - W(x + \mathrm{i}0)}{2\mathrm{i}\pi} = \frac{M(x)s(x)}{2\pi}\mathbb{1}_S(x).$$

By construction, W is the Stieltjes transform of μ. Since M has a single zero between each components of the support, μ is a positive measure – see the sign discussion in Section 5.2.3. Define $U(x) = V(x) - 2\int_S \ln|x-\xi|\mathrm{d}\mu(\xi)$. We clearly have

$$\forall x \in S \qquad U'(x) = V'(x) - W_1(x+\mathrm{i}0) - W_1(x-\mathrm{i}0) = 0.$$

Integrating this from a_h to $x \in (a_h, b_h)$, we find a constant c_h such that $U(x) = c_h$ for any $x \in (a_h, b_h)$. Besides, for any $h \in [g]$ we compute

$$c_h - c_{h-1} = \int_{b_{h-1}}^{a_h} U'(x)\mathrm{d}x = \int_{b_{h-1}}^{a_h} \big(V'(x) - 2W_1(x)\big)\mathrm{d}x = \int_{b_{h-1}}^{a_h} M(x)s(x)\mathrm{d}x.$$

Since we have chosen $(z_1, \ldots, z_g)$ so that this integral vanishes, c_h is independent of h. As a result, μ satisfies the characterisation of the equilibrium measure from Theorem 5.2.1 (unconstrained case). By uniqueness, this must be the equilibrium measure: $\mu = \mu_{\mathrm{eq}}$. □

5.3.5 Deformations of the curve

We consider real and complex deformations of the complex curves, that will be used in Section 5.4 to extend the validity of our formulae beyond their realisation for spectral curves of β-ensembles. We first show that within the class of spectral curves of β-ensembles, we can always realise any vector of filling fractions in a small neighborhood of a given one by perturbation of the support.

Lemma 5.3.12. *Let $a_0 < b_0 < \cdots < a_g < b_g$ and take a corresponding $M(x) = t_{2g+2}\prod_{h=1}^{g}(x - z_h)$ with $z_h \in (b_{h-1}, a_h)$ as in Proposition 5.3.11. There exists a small neighborhood $\Omega \subset \mathbb{R}^{2g}$ of $(a_h, z_h)_{h=1}^{g}$ such that the map $\Pi : \Omega \to \mathbb{R}^{2g}$ given by*

$$\Pi(\tilde{a}_1, \tilde{z}_1, \ldots, \tilde{a}_g, \tilde{z}_g) = \bigg(\int_{\tilde{a}_h}^{b_h} \tilde{M}(x)\sqrt{-\tilde{\sigma}(x)}\mathrm{d}x\,,\, \int_{b_{h-1}}^{\tilde{a}_h} \tilde{M}(x)\sqrt{\tilde{\sigma}(x)}\mathrm{d}x\bigg)_{h=1}^{g},$$

is a diffeomorphism onto its image, where we have set:

$$\tilde{M}(x) = t_{2g+2}\prod_{k=1}^{g}(x - \tilde{z}_k) \qquad \textit{and} \qquad \tilde{\sigma}(x) = (x-a_0)(x-b_0)\prod_{k=1}^{g}(x-\tilde{a}_k)(x-b_k).$$

This will be used in the following form.

Corollary 5.3.13. *There is a dense set of $a_0 < b_0 < \cdots < a_g < b_g$ for which there exists a β-ensemble whose associated equilibrium measure of Theorem 5.2.1 has filling fractions $\boldsymbol{\epsilon}^*$ whose components $\epsilon_1^*, \ldots, \epsilon_g^*$ are $\mathbb{Q}$-linearly independent.*

Proof of Lemma 5.3.12. Π is a smooth function of $(\tilde{a}_h, \tilde{z}_h)_{h=1}^{g}$ in the range $b_0 < \tilde{z}_1 < \tilde{a}_1 < b_1 < \tilde{z}_2 < \tilde{a}_2 < \cdots < \tilde{z}_g < \tilde{a}_g < b_g$. We compute its Jacobian

$$\begin{aligned}
&\det\begin{pmatrix} \int_{\tilde{a}_h}^{b_h} \frac{-\tilde{M}(x)\sqrt{-\tilde{\sigma}(x)}}{2(x-\tilde{a}_k)}\mathrm{d}x & \int_{\tilde{a}_h}^{b_h} \frac{-\tilde{M}(x)\sqrt{-\tilde{\sigma}(x)}}{(x-\tilde{z}_k)}\mathrm{d}x \\ \int_{b_{h-1}}^{\tilde{a}_h} \frac{-\tilde{M}(x)\sqrt{\tilde{\sigma}(x)}}{2(x-\tilde{a}_k)}\mathrm{d}x & \int_{b_{h-1}}^{\tilde{a}_h} \frac{-\tilde{M}(x)\sqrt{\tilde{\sigma}(x)}}{(x-\tilde{z}_k)}\mathrm{d}x \end{pmatrix}_{1\leq h,k\leq g} \\
&\qquad = \frac{1}{2^g}\int_{\tilde{a}_1}^{b_1}\cdots\int_{\tilde{a}_g}^{b_g}\prod_{h=1}^{g}\mathrm{d}x_h\tilde{M}(x_h)\sqrt{-\tilde{\sigma}(x_h)} \\
&\qquad\quad \times \int_{b_0}^{\tilde{a}_1}\cdots\int_{b_{g-1}}^{\tilde{a}_g}\mathrm{d}\xi_h\tilde{M}(\xi_h)\sqrt{\tilde{\sigma}(\xi_h)}\cdot\det\begin{pmatrix}\frac{1}{x_h-\tilde{a}_k} & \frac{1}{x_h-\tilde{z}_k} \\ \frac{1}{\xi_h-\tilde{a}_k} & \frac{1}{\xi_h-\tilde{z}_k}\end{pmatrix}_{1\leq h,k\leq g},
\end{aligned} \tag{5.29}$$

where we used the fact that $\sqrt{\pm\tilde{\sigma}(x)}$ vanishes at the endpoints of the integration intervals. The determinant in the integrand is a Cauchy determinant and can be readily evaluated

$$\det\begin{pmatrix} \frac{1}{x_h-\tilde{a}_k} & \frac{1}{x_h-\tilde{z}_k} \\ \frac{1}{\xi_h-\tilde{a}_k} & \frac{1}{\xi_h-\tilde{z}_k} \end{pmatrix}_{1\leq h,k\leq g} = \Delta(\tilde{\boldsymbol{a}})\Delta(\tilde{\boldsymbol{z}})\Delta(\boldsymbol{x})\Delta(\boldsymbol{\xi}) \prod_{h,k=1}^{g} \frac{(\tilde{z}_h-\tilde{a}_k)(\xi_h-x_k)}{(x_h-\tilde{a}_k)(\xi_h-\tilde{a}_k)(x_h-\tilde{z}_k)(\xi_h-\tilde{z}_k)}$$
$$= \frac{t_{2g+2}^{2g}\Delta(\tilde{\boldsymbol{a}})\Delta(\tilde{\boldsymbol{z}})\Delta(\boldsymbol{x})\Delta(\boldsymbol{\xi})}{\prod_{h=1}^{g}\tilde{M}(x_h)\tilde{M}(\xi_h)} \prod_{h,k=1}^{g} \frac{(\tilde{z}_h-\tilde{a}_k)(\xi_h-x_k)}{(x_h-\tilde{a}_k)(\xi_h-\tilde{a}_k)}. \tag{5.30}$$

For $(\tilde{a}_h, \tilde{z}_h)_{h=1}^{g}$ close enough to $(a_h, z_h)_{h=1}^{g}$, the zeros of $\tilde{M}$ are outside $\bigsqcup_{h=1}^{g}[\tilde{a}_h, \tilde{b}_h]$, so that the sign of the integrand in (5.29) remains constant in the whole integration range. The determinant of the Jacobian of Π is thus nonzero, and Π is a local diffeomorphism. □

Proof of Corollary 5.3.13. If $(\tilde{a}_h, \tilde{z}_h)_{h=1}^{g} \in \Omega$, call $\tilde{\mu}$ the measure supported on $\tilde{S} = [a_0, b_0] \cup \bigsqcup_{h=1}^{g}[\tilde{a}_h, b_h]$ with density $\frac{1}{2\pi}\tilde{M}(x)\sqrt{-\tilde{\sigma}(x)}$. At $(a_h, z_h)_{h=1}^{g}$ this $\tilde{\mu}$ is by construction the equilibrium measure of a β-ensemble, which we simply denote μ: it is in particular a probability measure with vector of filling fractions $(\epsilon_h^*)_{h=1}^{g}$ and the h-th second component of $\Pi(a_1, z_1, \ldots, a_g, z_g)$ is equal to $(-1)^{g-h}(U(a_h) - U(b_{h-1})) = 0$ for $h \in [g]$. So, Π induces a homeomorphism from Ω to a neighborhood $\Omega' \subset \mathbb{R}^{2g}$ of $\big((-1)^{g-h}2\pi\epsilon_h^*\,,\,0\big)_{h=1}^{g}$. By continuity with respect to the parameters, $\tilde{\mu}$ remains a positive measure on each component of $\tilde{S}$ for all parameters in a (possibly smaller) Ω, and that the total mass of $\tilde{\mu}$ defines a positive continuous function on Ω. In particular, $\tilde{\mu}' = \tilde{\mu}/\tilde{\mu}(\tilde{S})$ is a probability measure on $\tilde{S}$.

If $\epsilon_1^*, \ldots, \epsilon_g^*$ are $\mathbb{Q}$-linearly dependent, we can approximate $\big((-1)^{g-h}2\pi\epsilon_h^*\,,\,0\big)_{h=1}^{g}$ to arbitrary precision by $2g$-tuples $\big((-1)^{g-h}2\pi\tilde{\epsilon}_h\,,\,0\big)_{h=1}^{g} \in \Omega'$ such that $\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_g$ are $\mathbb{Q}$-linearly independent. Applying Π^{-1}, we get an approximation $(\tilde{a}_h, \tilde{z}_h)_{h=1}^{g}$ of $(a_h, z_h)_{h=1}^{g}$ at arbitrary precision whose associated probability measure $\tilde{\mu}'$ is by construction (follow the proof of Proposition 5.3.11) the equilibrium measure of the β-ensemble with potential

$$\tilde{V}(x) = \frac{1}{\tilde{\mu}(\tilde{S})}\int_0^x \mathcal{V}[\tilde{M}\cdot\tilde{s}](x),$$

with $\tilde{s}$ like s of Section 5.2.3 but with $\tilde{a}$s instead of as, i.e. a choice of square root of $\prod_{h=0}^{g}(x-\tilde{a}_h)(x-b_h)$. Let us detail this claim. We introduce the effective energy $\tilde{U}(x) = \tilde{V}(x) - 2\int_{\tilde{S}} \ln|x-\xi|\mathrm{d}\tilde{\mu}(\xi)$ associated to $\tilde{\mu}$. It satisfies both $\tilde{U}(\tilde{a}_h) - \tilde{U}(b_h) = 0$ (because the second component of the image of $(\tilde{a}_h, \tilde{z}_h)_{h=1}^{g}$ by Π is zero) and $\tilde{U}'(x) = 0$ for x in the support of $\tilde{\mu}$ (because the Stieltjes transform of $\tilde{V}'$ is the polynomial part of the density of $\tilde{\mu}$, up to a factor 2π). It satisfies (5.4), and is the equilibrium measure of the β-ensemble with potential $\tilde{V}$. This equilibrium measure has vector of filling fractions $(\tilde{\epsilon}_h/\tilde{\mu}(\tilde{S}))_{h=1}^{g}$, whose components remain $\mathbb{Q}$-linearly independent. □

In a second step, we will leave the realm of spectral curves of β-ensembles and rather consider their complex deformations. Here it becomes important to keep track of the marking. The equation of a hyperelliptic curve $s^2 = \prod_{h=0}^{g}(x-a_h)(x-b_h)$ is parameterised by the set $\mathbf{\Delta}_{2g+2}$ of $(2g+2)$-tuple $(a_h, b_h)_{h=0}^{g}$ of pairwise distinct complex numbers. Its universal cover $\widetilde{\mathbf{\Delta}}_{2g+2}$ based at a tuple of strictly increasing real numbers parametrises the equation of the hyperelliptic curve together with a choice of marking: at the base point it is the one described in Section 5.3.3, and there is a unique way to get from there a marking for any other point in $\widetilde{\mathbf{\Delta}}_{2g+2}$ by performing continuous deformations of the representatives of the homology cycles. The outcome is an analytic family $\hat{\boldsymbol{C}} \to \widetilde{\mathbf{\Delta}}_{2g+2}$ of marked hyperelliptic curves, which coincide with the one described in Section 5.3.3 above the connected component of the base point in the real locus of $\widetilde{\mathbf{\Delta}}_{2g+2}$. Concretely, in other real connected components, the symplectic basis of homology has changed by an $\mathrm{Sp}_{2g}(\mathbb{Z})$-transformation compared to Section 5.3.3, and so must have the matrix of periods. Let us denote likewise $\widetilde{\boldsymbol{C}}$ the family of universal covers over $\widetilde{\mathbf{\Delta}}_{2g+2}$. We will rely on the following basic fact in complex geometry — see e.g. [CMP17, Chapter 1].

Lemma 5.3.14. *The period matrix (5.17) is a holomorphic function on $\widetilde{\mathbf{\Delta}}_{2g+2}$. The Abel map based at ∞_+ is a holomorphic function $\widetilde{C} \to \mathbb{C}^g$.*

5.4 Asymptotics of the partition function and the kernels

5.4.1 Expansion of the partition function and generalised central limit theorem

The large N asymptotic expansion of the partition function of the β-ensembles in the multi-cut regime was established in [BG24], under assumptions which are satisfied for the potentials that we consider in Theorem 5.2.1. In particular, the off-criticality assumption on A corresponds to M of Lemma 5.2.3 having no zeros on A. We reproduce here the formulae for these asymptotics, which will be our starting point.

Theorem 5.4.1. *[BG24, Theorem 1.5] Let $g \geq 1$, let V as in Theorem 5.2.1 and assume M from Lemma 5.2.3 has no zeros on A. The partition function has the following expansion as $N \to \infty$*

$$Z_N^V \sim N^{\frac{\beta}{2}N+\varkappa} e^{N^2\mathcal{E}[\mu_{eq}]+N\mathcal{S}[\mu_{eq}]+\mathcal{G}[\mu_{eq}]}\,\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{eq}\big|\tfrac{\beta}{2}\boldsymbol{\tau}\big).$$

Here

$$\mathcal{S}[\mu] = \left(1-\frac{\beta}{2}\right)\big(\mathrm{Ent}[\mu]-\ln(\beta/2)\big)+(\beta/2)\ln(2\pi/e)-\ln\Gamma(\beta/2),$$

where $\mathrm{Ent}[\mu]$ is the von Neumann entropy of the probability measure μ, $\varkappa$ is a known universal constant depending only on g and β, $\mathcal{G}[\mu]$ is a continuous functional whose expression is irrelevant for our purposes, and

$$\boldsymbol{v}_{eq} = \frac{\nabla_{\epsilon}\mathcal{S}[\mu_{eq,\boldsymbol{\epsilon}}]}{2\mathrm{i}\pi}\bigg|_{\boldsymbol{\epsilon}=\boldsymbol{\epsilon}^*}. \tag{5.31}$$

In (5.31) one differentiates with respect to $\boldsymbol{\epsilon} = (\epsilon_1,\ldots,\epsilon_g)$, keeping in mind that the filling fraction ϵ_0, associated to the component of the support $[a_0,b_0]$, satisfies $\epsilon_0 = 1-(\epsilon_1+\cdots+\epsilon_g)$ and thus depends on $\epsilon_1,\ldots,\epsilon_g$. The kernels appearing in the determinantal and pfaffian formulae of Borodin and Strahov (Theorems 5.2.6–5.2.8) can be estimated using the following generalised central limit theorem. We use the notation $\oint_S$ for a sum of contour integrals in the positive direction around the connected components of the support S of the equilibrium measure.

Theorem 5.4.2 ([BG24, Theorem 1.6])**.** *Let f be a holomorphic function in a complex neighborhood of A. Under the same assumptions as Theorem 5.4.1, we have as $N \to \infty$:*

$$\Big\langle e^{\sum_{i=1}^N f(\lambda_i)}\Big\rangle_N^V \sim e^{N\mathcal{L}[f]+\mathcal{H}[f]+\frac{1}{2}\mathcal{Q}[f,f]}\,\frac{\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{eq}+\boldsymbol{\mathcal{U}}[f]\big|\frac{\beta}{2}\boldsymbol{\tau}\big)}{\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{eq}\big|\frac{\beta}{2}\boldsymbol{\tau}\big)}.$$

Here

$$\mathcal{L}[f] = \oint_S W_1(\xi)\frac{f(\xi)\mathrm{d}\xi}{2\mathrm{i}\pi},\qquad \mathcal{Q}[f,f] = \frac{2}{\beta}\oint_{S^2} W_2(\xi_1,\xi_2)\frac{f(\xi_1)\mathrm{d}\xi_1}{2\mathrm{i}\pi}\frac{f(\xi_2)\mathrm{d}\xi_2}{2\mathrm{i}\pi},$$

where W_1 and W_2 are calculated in a model with fixed filling fraction tending to $\boldsymbol{\epsilon}^*$, $\mathcal{H}[f]$ is a linear form whose expression is irrelevant, and

$$\boldsymbol{\mathcal{U}}[f] = \oint_S f(X(z))\frac{\mathrm{d}\boldsymbol{u}(z)}{2\mathrm{i}\pi}.$$

5.4.2 Three explicit formulae

We will establish in Section 5.5.4 the following expression for the equilibrium energy in terms of the geometry of the spectral curve.

Proposition 5.4.3. *The equilibrium energy is*

$$-\mathcal{E}[\mu_{eq}] = \frac{\beta}{2}\mathcal{L}[V] + \frac{\beta^2}{8}\mathcal{Q}[V,V] + \mathrm{i}\pi\beta\boldsymbol{\epsilon}^* \cdot \big(\boldsymbol{\tau}(\boldsymbol{\epsilon}^*) + \boldsymbol{u}(\infty_-)\big).$$

The right-hand side is proportional to $\frac{\beta}{2}$, since W_2 contains a factor $\frac{2}{\beta}$, and so is the left-hand side. Accordingly this is an identity between β-independent quantities. There is a classical link between random matrix theory and the theory of Frobenius manifolds: the free energy at leading order, namely $\mathcal{E}[\mu_{\mathrm{eq}}]$ coincides with the prepotential of the Hurwitz–Frobenius manifold associated to the spectral curve of the random matrix ensemble. A formula for this free energy was established in the more general context of the two matrix model in [Ber03], involving only the geometry of the spectral curve. It involves the same ingredient but does not have exactly the same form as Proposition 5.4.3, which is the formula we need.

Before going further, we give two extra formulae. The first one evaluates the argument $\boldsymbol{v}_{\mathrm{eq}}$ of the theta functions in Theorems 5.4.1-5.4.2; it is not necessary for Section 5.5, but we include it for completeness. The second one will be used in the proof of Lemma 5.4.8.

Proposition 5.4.4. *Assume M in (5.7) is of the form $M(x) = t_{2g+2}\prod_{h=1}^{g}(x - z_h)$ with roots $z_h \notin A$ satisfying $b_{h-1} < z_h < a_h$ for any $h \in [g]$, and denote by $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_g)$ the canonical basis of $\mathbb{C}^g$.*
The function Im $\boldsymbol{u}(z) = \int_{\infty_+}^{z} \mathrm{d}\boldsymbol{u}$ is a single-valued function of $z \in \hat{C}_+$, and we have

$$\boldsymbol{v}_{eq} = 2\pi\left(1 - \frac{\beta}{2}\right)\left[\frac{g+1}{2\mathrm{i}}\boldsymbol{u}(\infty_-) + \sum_{k=1}^{g}\left(Im\,\boldsymbol{u}(z_k) + \frac{g+1-k}{2\mathrm{i}}\boldsymbol{\tau}(e_k)\right)\right]. \tag{5.32}$$

Proof. See Appendix 5.6. As we explain in Section 5.5.4, $\boldsymbol{u}(\infty_-)$ and $\boldsymbol{\tau}$ are purely imaginary, so all terms in the right-hand side are real, as it should be. The domain $\hat{C}_+$ is homeomorphic to the non simply-connected domain $\hat{\mathbb{C}} \setminus S$, so $\int_{\infty_+}^{z} \mathrm{d}\boldsymbol{u}$ is multi-valued. However, the ambiguities are $\mathcal{A}$-periods of $\mathrm{d}\boldsymbol{u}$ which are real, so Im $\boldsymbol{u}(z)$ is single-valued. ☐

Remark 5.4.5. If we remove the assumption on the roots of M in Lemma 5.4.4, we can still compute $\boldsymbol{v}_{\mathrm{eq}}$ and obtain a formula of a similar form.

Lemma 5.4.6.

$$\boldsymbol{\mathcal{U}}[V] = -\operatorname{Res}_{\infty_+} V\mathrm{d}\boldsymbol{u} = \boldsymbol{\tau}(\boldsymbol{\epsilon}^*) + \boldsymbol{u}(\infty_-).$$

Proof. The first equality is obtained by moving the contour around S to ∞_+. We focus on the second equality. Since $\mathrm{d}V = 2(\phi - Y\mathrm{d}X)$, we have

$$\operatorname{Res}_{\infty_+} V\mathrm{d}\boldsymbol{u} = -\operatorname{Res}_{\infty_+} \boldsymbol{u}\mathrm{d}V = -2\operatorname{Res}_{\infty_+} \boldsymbol{u}(\phi - Y\mathrm{d}X)\,.$$

Since ∞_+ is the base point for the Abel map, we have $\boldsymbol{u}(\infty_+) = 0$ and since ϕ has only a simple pole at ∞_+, the first term gives a vanishing residue. The hyperelliptic involution preserves X, sends Y and $\mathrm{d}\boldsymbol{u}$ to their opposite, and ∞_+ to ∞_-. Hence, it sends $\boldsymbol{u}$ to $\boldsymbol{u}(\infty_-) - \boldsymbol{u}$, where $\infty_- \in \hat{C}^0$. Using the involution as a change of variables, we get

$$\operatorname{Res}_{\infty_+} \boldsymbol{u}Y\mathrm{d}X = \operatorname{Res}_{\infty_-}(\boldsymbol{u} - \boldsymbol{u}(\infty_-))Y\mathrm{d}X = \operatorname{Res}_{\infty_-} \boldsymbol{u}Y\mathrm{d}X - \boldsymbol{u}(\infty_-).$$

For the last equality, since the only poles of $Y\mathrm{d}X$ are $\infty_\pm$ we could evaluate

$$\operatorname{Res}_{\infty_-} Y\mathrm{d}X = -\operatorname{Res}_{\infty_+} Y\mathrm{d}X = -\operatorname{Res}_{\infty_+} \phi = 1.$$

We then write

$$\operatorname{Res}_{\infty_+} V\mathrm{d}\boldsymbol{u} = \big(\operatorname{Res}_{\infty_+} + \operatorname{Res}_{\infty_-}\big)Y\mathrm{d}X - \boldsymbol{u}(\infty_-).$$

The first term can be computed with the Riemann bilinear identity [FK92, Equation 3.0.2]

$$\big(\mathrm{Res}_{\infty_+}+\mathrm{Res}_{\infty_-}\big)\boldsymbol{u}Y\mathrm{d}X=\frac{1}{2\mathrm{i}\pi}\sum_{h=1}^{g}\bigg(\oint_{\mathcal{A}_h}\mathrm{d}\boldsymbol{u}\cdot\oint_{\mathcal{B}_h}Y\mathrm{d}X-\oint_{\mathcal{B}_h}\mathrm{d}\boldsymbol{u}\cdot\oint_{\mathcal{A}_h}Y\mathrm{d}X\bigg). \tag{5.33}$$

Taking into account that for any $h,k\in[g]$, we have $\oint_{\mathcal{B}_h}Y\mathrm{d}X=0$ (Remark 5.3.8), and $\oint_{\mathcal{B}_h}\mathrm{d}u_k=\tau_{h,k}$ and $\oint_{\mathcal{A}_h}Y\mathrm{d}X=2\mathrm{i}\pi\,\epsilon_h^*$, we find that (5.33) is equal to $-\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)$. ☐

5.4.3 Kernel asymptotics: intermediate computations

We need to compute an asymptotic equivalent as $K\to\infty$ of kernels of the form:

$$\frac{Z_M^{\frac{K}{M}V}}{Z_K^V}\left\langle\prod_{j=1}^{m}\det(x_j-\Lambda)^{c_j}\right\rangle_M^{\frac{K}{M}V},$$

where $x_1,\ldots,x_m\notin A$, $c_1,\ldots,c_m\in\mathbb{Z}$, and $(M-K)=p$ is a fixed integer. Notice that

$$\frac{Z_M^{\frac{K}{M}V}}{Z_K^V}\left\langle\prod_{j=1}^{m}\det(x_j-\Lambda)^{c_j}\right\rangle_M^{\frac{K}{M}V}=\frac{Z_M^V}{Z_K^V}\left\langle e^{\sum_{i=1}^{M}f_{\boldsymbol{c}}(\lambda_i)+\frac{\beta}{2}pV(\lambda_i)}\right\rangle_M^V, \tag{5.34}$$

where we used the holomorphic function on a neighborhood of A

$$f_{\boldsymbol{c}}(\lambda)=\sum_{j=1}^{m}c_j\ln(x_j-\lambda),$$

and for $x_j\notin A$ we choose the cut of the logarithm away from A. In order to access the asymptotics of (5.34) via Theorem 5.4.2, we first have to evaluate the following quantities

Lemma 5.4.7. *Let $z,z_1,z_2\in C_+$. We have:*

$$\begin{aligned}\mathcal{L}\big[\ln(X(z)-\bullet)\big]&=2\mathrm{i}\pi\boldsymbol{\epsilon}^*\cdot\boldsymbol{u}(z)-\ln\big(\eta(z)E(z,\infty_+)^2\mathrm{d}\zeta_{\infty_+}(\infty_+)\big)\\&\qquad-\sum_{k=1}^{d}\frac{t_k}{k}\int_{\infty_+}^{z}\mathrm{d}B_{\infty_-,k},\\ \mathcal{Q}\big[\ln(X(z)-\bullet),V\big]&=\frac{2}{\beta}\sum_{k=1}^{d}\frac{t_k}{k}\int_{\infty_+}^{z}\mathrm{d}B_{\infty_+,k},\\ \mathcal{Q}\big[\ln(X(z_1)-\bullet),\ln(X(z_2)-\bullet)\big]&=\frac{2}{\beta}\ln\left(\frac{E(z_1,z_2)}{E(z_1,\infty_+)E(z_2,\infty_+)(X(z_2)-X(z_1))\mathrm{d}\zeta_{\infty_+}(\infty_+)}\right),\\ \mathcal{Q}\big[\ln(X(z)-\bullet),\ln(X(z)-\bullet)\big]&=\frac{2}{\beta}\ln\left(\frac{1}{E(z,\infty_+)^2\mathrm{d}\zeta_{\infty_+}(\infty_+)}\right),\end{aligned}$$

where $\eta(z)$ is defined in (5.37). In particular, we observe the simplification

$$\mathcal{L}\big[\ln(X(z)-\bullet)\big]+\frac{\beta}{2}\mathcal{Q}\ln\big[\ln(X(z)-\bullet),V\big]=2\mathrm{i}\pi\boldsymbol{\epsilon}^*\cdot\boldsymbol{u}(z)-\ln\big(\eta(z)E(z,\infty_+)^2\mathrm{d}\zeta_{\infty_+}(\infty_+)\big). \tag{5.35}$$

Proof. Let $x=X(z)$. The first two formulas are a consequence of the decomposition (5.24) of ϕ. We have:

$$\begin{aligned}\mathcal{L}\big[\ln(x-\bullet)\big] &= \oint_S \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\ln(x-\xi)W_1(\xi)\\ &= \oint_S \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\bigg[\int_\infty^x \Big(\frac{1}{\xi'-\xi}-\frac{1}{\xi'}\Big)\mathrm{d}\xi' + \ln x\bigg]\,W_1(\xi)\\ &= \ln(x) + \int_\infty^x \mathrm{d}\xi'\bigg(\oint_S \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{W_1(\xi)}{\xi'-\xi} - \frac{1}{\xi'}\bigg)\\ &= \ln(x) + \int_\infty^x \mathrm{d}\xi'\bigg(W_1(\xi') - \frac{1}{\xi'}\bigg)\\ &= \ln(x) + \int_{\infty_+}^z \Big(\phi - \frac{\mathrm{d}X}{X}\Big).\end{aligned}$$

We can expand this as

$$\begin{aligned}\mathcal{L}\big[\ln(x-\bullet)\big] &= \ln x + 2\mathrm{i}\pi\boldsymbol{\epsilon}^*\cdot\boldsymbol{u}(z) + \int_{\infty_+}^z \Big(\mathrm{d}S_{\infty_+,\infty_-} + \frac{\mathrm{d}\zeta_{\infty_+}}{\zeta_{\infty_+}}\Big) - \sum_{k=1}^d \frac{t_k}{k}\int_{\infty_+}^z \mathrm{d}B_{\infty_-,k}\\ &= \ln x + 2\mathrm{i}\pi\boldsymbol{\epsilon}^*\cdot\boldsymbol{u}(z) + \int_{\infty_+}^z \mathrm{d}_{z'}\ln\bigg(\frac{E(z',\infty_-)\zeta_{\infty_+}(z')}{E(z',\infty_+)}\bigg) - \sum_{k=1}^d \frac{t_k}{k}\int_{\infty_+}^z B_{\infty_-,k}\\ &= \ln x + 2\mathrm{i}\pi\boldsymbol{\epsilon}^*\cdot\boldsymbol{u}(z) + \ln\bigg(\frac{E(z,\infty_-)\zeta_{\infty_+}(z)}{E(z,\infty_+)E(\infty_+,\infty_-)\mathrm{d}\zeta_{\infty_+}(\infty_+)}\bigg) - \sum_{k=1}^d \frac{t_k}{k}\int_{\infty_+}^z B_{\infty_-,k}.\end{aligned} \tag{5.36}$$

and $\ln x$ cancels with $\ln\zeta_{\infty_+}(z)$. We introduce the 1-form on $\widetilde{C}$

$$\eta(z) = \frac{E(\infty_+,\infty_-)}{E(z,\infty_+)E(z,\infty_-)} \tag{5.37}$$

and use it to get rid of ∞_- in (5.36). This leads to the claimed formula.

For the second formula

$$\mathcal{Q}\big[\ln(x-\bullet),V\big] = \frac{2}{\beta}\oint_{S^2}\frac{\mathrm{d}\xi_1}{2\mathrm{i}\pi}\frac{\mathrm{d}\xi_2}{2\mathrm{i}\pi}\,V(\xi_1)\bigg[\ln x + \int_{\infty_+}^x \Big(\frac{1}{\xi'-\xi_2}-\frac{1}{\xi'}\Big)\mathrm{d}\xi'\bigg]W_2(\xi_1,\xi_2).$$

We move the ξ_2-contour to ∞. Since W_2 has no residue at ∞ the first and third term disappear and we get:

$$\begin{aligned}\mathcal{Q}\big[\ln(x-\bullet),V\big] &= \frac{2}{\beta}\oint_S \frac{\mathrm{d}\xi_1}{2\mathrm{i}\pi}V(\xi_1)\int_{\infty_+}^x W_2(\xi_1,\xi')\mathrm{d}\xi'\\ &= \frac{2}{\beta}\oint_S \frac{\mathrm{d}\xi_1}{2\mathrm{i}\pi}V(\xi_1)\int_{\infty_+}^x \Big(W_2(\xi_1,\xi') + \frac{1}{(\xi_1-\xi')^2}\Big)\mathrm{d}\xi',\end{aligned}$$

where in the second line the shift does not affect the integral around S as it is holomorphic near S. Then, we write $X(w)=\xi_1$ and $X(w')=\xi'$, consider these integrals as integrals on $\widetilde{C}$, and recognise via (5.27) the fundamental bidifferential B. Since the path on which we integrate w' remains in the first sheet away from the cut, we can move the integral over w to surround the pole ∞_- of $V(X(w))$. We get:

$$\begin{aligned}\mathcal{Q}\big[\ln(x-\bullet),V\big] &= \frac{2}{\beta}\int_{w'=\infty_+}^z \operatorname*{Res}_{w=\infty_-}\mathrm{d}X(w)\,V(X(w))\,B(w,w')\\ &= \frac{2}{\beta}\sum_{k=1}^d\frac{t_k}{k}\int_{w'=\infty_+}^z \operatorname*{Res}_{w=\infty_-}\zeta_{\infty_-}(w)^{-k}B(w,w')\\ &= \frac{2}{\beta}\sum_{k=1}^d\frac{t_k}{k}\int_{\infty_+}^z \mathrm{d}B_{\infty_-,k},\end{aligned}$$

as desired.

For the third formula, let $x_i = X(z_i)$. Using Section 5.3.3, we find

$$\begin{aligned}\mathcal{Q}\big[\ln(x_1-\bullet),\ln(x_2-\bullet)\big] &= \frac{2}{\beta}\oint_{S^2}\frac{\mathrm{d}\xi_1\mathrm{d}\xi_2}{(2\mathrm{i}\pi)^2}\,W_2(\xi_1,\xi_2)\ln(x_1-\xi_1)\ln(x_2-\xi_2)\\ &= \frac{2}{\beta}\int_{\infty_+}^{z_1}\int_{\infty_+}^{z_2}\mathcal{W}_2(w_1,w_2)\mathrm{d}X(w_1)\mathrm{d}X(w_2)\end{aligned}$$

after we handle the logarithms like in the previous proofs. Then:

$$\begin{aligned}\mathcal{Q}\big[\ln(x_1-\bullet),\ln(x_2-\bullet)\big] &= \frac{2}{\beta}\int_{\infty_+}^{z_1}\int_{\infty_+}^{z_2}\bigg(B(w_1,w_2)-\frac{\mathrm{d}X(w_1)\mathrm{d}X(w_2)}{(X(w_1)-X(w_2))^2}\bigg)\\ &= \frac{2}{\beta}\ln\bigg(\frac{E(z_1,z_2)E_0(z_1,\infty_+)E_0(\infty_+,z_2)}{E(z_1,\infty_+)E(\infty_+,z_2)E_0(z_1,z_2)}\bigg).\end{aligned}$$

where we used Lemma 5.3.5 both for $\widetilde{C}$ and $\hat{\mathbb{C}}$, and (5.20) to get rid of the ratio of the relative prime form with two arguments ∞_+. Since the prime forms are antisymmetric in their two variables, we can arrange the formula to have ∞_+ always in the second argument. The presence of ∞_+ in E_0 factors can be understood by first replacing it by a point $\tilde{z}$, and then letting $\tilde{z}\to\infty_+$. Due to Lemma 5.3.6 the product of the two E_0-factors involving ∞_+ only gives a sign when we use the local coordinate ζ_{∞_+}, and using $E_0(z_1,z_2) = (X(z_1)-X(z_2))/\sqrt{\mathrm{d}X(z_1)\mathrm{d}X(z_2)}$ leads to the claim. □

We are now in position to evaluate the asymptotics of the kernels. We will mainly be interested in a situation with only two variables:

$$\begin{aligned}\mathcal{K}_M^{\frac{K}{M}V}\big(\begin{smallmatrix} c & \tilde{c}\\ x & \tilde{x}\end{smallmatrix}\big) &:= \Big\langle \det(x-\Lambda)^c\det(\tilde{x}-\Lambda)^{\tilde{c}}e^{\frac{\beta}{2}p\mathrm{Tr}V(\Lambda)}\Big\rangle_M^V\\ &= \Big\langle e^{\sum_{i=1}^M c\ln(x-\lambda_i)+\tilde{c}\ln(\tilde{x}-\lambda_i)+\frac{\beta}{2}pV(\lambda_i)}\Big\rangle_M^V.\end{aligned} \tag{5.38}$$

where $p = M-K$. It will be used in the form:

$$\frac{Z_M^{\frac{K}{M}V}}{Z_K^V}\Big\langle\det(x-\Lambda)^c\det(\tilde{x}-\Lambda)^{\tilde{c}}\Big\rangle_M^{\frac{K}{M}V} = \frac{Z_M^V}{Z_K^V}\,\mathcal{K}_M^{\frac{K}{M}V}\big(\begin{smallmatrix} c & \tilde{c}\\ x & \tilde{x}\end{smallmatrix}\big). \tag{5.39}$$

Lemma 5.4.8. *Let $z,\tilde{z}\in C_+$ and $c,\tilde{c}\in\mathbb{Z}$. We have as $K\to\infty$ and p is a fixed integer:*

$$\begin{aligned}&\mathcal{K}_{K+p}^{\frac{K}{K+p}V}\big(\begin{smallmatrix} c & \tilde{c}\\ X(z) & X(\tilde{z})\end{smallmatrix}\big)\\ &\sim e^{Kc\mathcal{L}[\ln(X(z)-\bullet)]+K\tilde{c}\mathcal{L}[\ln(X(\tilde{z})-\bullet)]+c\mathcal{H}[\ln(X(z)-\bullet)]+\tilde{c}\mathcal{H}[\ln(X(\tilde{z})-\bullet)]+K\frac{\beta}{2}p\mathcal{L}[V]+\frac{\beta}{2}p^2\mathcal{L}[V]+\frac{\beta}{2}p\mathcal{H}[V]+\frac{\beta^2}{8}p^2\mathcal{Q}[V,V]}\\ &\quad\times e^{2\mathrm{i}\pi p\boldsymbol{\epsilon}^*\cdot(c\boldsymbol{u}(z)+\tilde{c}\boldsymbol{u}(\tilde{z}))}\Big(\eta(z)E(z,\infty_+)^2\mathrm{d}\zeta_{\infty_+}(\infty_+)\Big)^{-pc}\Big(\eta(\tilde{z})E(\tilde{z},\infty_+)^2\mathrm{d}\zeta_{\infty_+}(\infty_+)\Big)^{-p\tilde{c}}\\ &\quad\times e^{\frac{1}{2}c^2\mathcal{Q}[\ln(X(z)-\bullet),\ln(X(z)-\bullet)]+\frac{1}{2}\tilde{c}^2\mathcal{Q}[\ln(X(\tilde{z})-\bullet),\ln(X(\tilde{z})-\bullet)]}\\ &\quad\times\bigg(\frac{E(z,\tilde{z})}{E(z,\infty_+)E(\tilde{z},\infty_+)(X(\tilde{z})-X(z))\mathrm{d}\zeta_{\infty_+}(\infty_+)}\bigg)^{\frac{2}{\beta}c\tilde{c}}\\ &\quad\times\frac{\vartheta_{-(K+p)\boldsymbol{\epsilon}^*,\boldsymbol{0}}\big(\boldsymbol{v}_{eq}+c\boldsymbol{u}(z)+\tilde{c}\boldsymbol{u}(\tilde{z})+\frac{\beta}{2}p(\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)+\boldsymbol{u}(\infty_-))\big|\frac{\beta}{2}\boldsymbol{\tau}\big)}{\vartheta_{-M\boldsymbol{\epsilon}^*,\boldsymbol{0}}\big(\boldsymbol{v}_{eq}\big|\frac{\beta}{2}\boldsymbol{\tau}\big)}.\end{aligned}$$

In particular if $p = 0$, we have as $M \to \infty$

$$
\begin{aligned}
\mathcal{K}_M^V\left(\begin{smallmatrix} c & \tilde{c} \\ X(z) & X(\tilde{z}) \end{smallmatrix}\right) \sim\ & e^{Mc\mathcal{L}[\ln(X(z)-\bullet)]+M\tilde{c}\mathcal{L}[\ln(X(\tilde{z})-\bullet)]+c\mathcal{H}[\ln(X(z)-\bullet)]+\tilde{c}\mathcal{H}[\ln(X(\tilde{z})-\bullet)]} \\
& \times e^{\frac{1}{2}c^2\mathcal{Q}[\ln(X(z)-\bullet),\ln(X(z)-\bullet)]+\frac{1}{2}\tilde{c}^2\mathcal{Q}[\ln(X(\tilde{z})-\bullet),\ln(X(\tilde{z})-\bullet)]} \\
& \times \left(\frac{E(z,\tilde{z})}{(X(\tilde{z})-X(z))E(z,\infty_+)E(\tilde{z},\infty_+)\mathrm{d}\zeta_{\infty_+}(\infty_+)}\right)^{\frac{2}{\beta}c\tilde{c}} \\
& \times \frac{\vartheta_{-M\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{eq}+c\boldsymbol{u}(z)+\tilde{c}\boldsymbol{u}(\tilde{z})\big|\frac{\beta}{2}\boldsymbol{\tau}\big)}{\vartheta_{-(K+p)\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{eq}\big|\frac{\beta}{2}\boldsymbol{\tau}\big)}.
\end{aligned}
$$

Proof. Let $x = X(z)$ and $\tilde{x} = X(\tilde{z})$. Applying Theorem 5.4.2 to the definition (5.38), we get as $M, K \to \infty$ while $M - K = p$ is fixed:

$$
\begin{aligned}
\mathcal{K}_M^{\frac{K}{M}V}\left(\begin{smallmatrix} c & \tilde{c} \\ x & \tilde{x} \end{smallmatrix}\right) \sim\ & \frac{Z_M^V}{Z_M^{\frac{K}{M}V}} e^{Mc\mathcal{L}[\ln(x-\bullet)]+M\tilde{c}\mathcal{L}[\ln(\tilde{x}-\bullet)]+c\mathcal{H}[\ln(x-\bullet)]+\tilde{c}\mathcal{H}[\ln(\tilde{x}-\bullet)]+M\frac{\beta}{2}p\mathcal{L}[V]+\frac{\beta}{2}\mathcal{H}[V]+\frac{\beta^2}{8}p^2\mathcal{Q}[V,V]} \\
& \times e^{\frac{1}{2}c^2\mathcal{Q}[\ln(x-\bullet),\ln(x-\bullet)]+c\tilde{c}\mathcal{Q}[\ln(x-\bullet),\ln(\tilde{x}-\bullet)]+\frac{1}{2}\tilde{c}^2\mathcal{Q}[\ln(\tilde{x}-\bullet),\ln(\tilde{x}-\bullet)]+\frac{\beta}{2}pc\mathcal{Q}[\ln(x-\bullet),V]+\frac{\beta}{2}p\tilde{c}\mathcal{Q}[\ln(\tilde{x}-\bullet),V]} \\
& \times \frac{\vartheta_{-M\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}+\frac{1}{2\mathrm{i}\pi}\oint_S\big[c\ln(x-\bullet)+\tilde{c}\ln(\tilde{x}-\bullet)+\frac{\beta}{2}pV\big]\mathrm{d}\boldsymbol{u}\big|\frac{\beta}{2}\boldsymbol{\tau}\big)}{\vartheta_{-M\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}\big|\frac{\beta}{2}\boldsymbol{\tau}\big)}.
\end{aligned}
$$

We split $M = K + p$ in the exponential and combine the newly created p-terms with the $\frac{\beta}{2}\mathcal{Q}[\ln, V]$ terms which is evaluated thanks to (5.35). We also replace the term $c\tilde{c}\mathcal{Q}[\ln,\ln]$ by its evaluation from Lemma 5.4.7, but refrain from doing so for the c^2 and the $\tilde{c}^2$ terms. Finally, writing the logarithm in the arguments of the theta function as a primitive, we get an expression in terms of the Abel map (a similar manipulation was carried out in the proof of Lemma 5.4.4), and Lemma 5.4.6 tells us $\frac{1}{2\mathrm{i}\pi}\oint_S V\mathrm{d}\boldsymbol{u} = \boldsymbol{\tau}(\boldsymbol{\epsilon}^*) + \boldsymbol{u}(\infty_-)$ which we need to multiply by $\frac{\beta p}{2}$ in the last argument of the theta function. Together with Lemma 5.4.6 this implies the claimed formula. □

The ratio of partition functions appearing in (5.39) will only be needed through the following combination.

Lemma 5.4.9. *For fixed $p \in \mathbb{Z}$, we have as $K \to \infty$*

$$
\frac{Z_{K+p}^V Z_{K-p}^V}{(Z_K^V)^2} \sim e^{2p^2\mathcal{E}[\mu_{eq}]+\mathrm{i}\pi\beta p^2\boldsymbol{\epsilon}^*\cdot\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)} \frac{\vartheta_{-K\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{eq}-\frac{\beta}{2}p\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)\big|\frac{\beta}{2}\boldsymbol{\tau}\big)\vartheta_{-K\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{eq}+\frac{\beta}{2}p\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)\big|\frac{\beta}{2}\boldsymbol{\tau}\big)}{\vartheta_{-K\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{eq}\big|\frac{\beta}{2}\boldsymbol{\tau}\big)^2}.
$$

Proof. Using Theorem 5.4.1 we find:

$$
\frac{Z_{K+p}^V}{Z_K^V} \sim e^{(2Kp+p^2)\mathcal{E}[\mu_{\mathrm{eq}}]+p\mathcal{S}[\mu_{\mathrm{eq}}]+\frac{\beta}{2}p(\ln K+1)} \frac{\vartheta_{-(K+p)\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}\big|\frac{\beta}{2}\boldsymbol{\tau}\big)}{\vartheta_{-K\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}\big|\frac{\beta}{2}\boldsymbol{\tau}\big)}.
$$

Multiplying this expression and the same with $p \to -p$ yields:

$$
\frac{Z_{K+p}^V Z_{K-p}^V}{(Z_K^V)^2} \sim e^{2p^2\mathcal{E}[\mu_{\mathrm{eq}}]} \frac{\vartheta_{-(K+p)\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}\big|\frac{\beta}{2}\boldsymbol{\tau}\big)\vartheta_{-(K-p)\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}\big|\frac{\beta}{2}\boldsymbol{\tau}\big)}{\vartheta^2_{-K\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}\big|\frac{\beta}{2}\boldsymbol{\tau}\big)}. \tag{5.40}
$$

We would like to rewrite all theta functions with a characteristic $-K\boldsymbol{\epsilon}^*$ instead of $-(K \pm p)\boldsymbol{\epsilon}^*$. For this, we come back to the definitions in Section 5.3.1 and find:

$$
\vartheta_{-(K+p)\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}\big|\tfrac{\beta}{2}\boldsymbol{\tau}\big) = e^{\mathrm{i}\pi\frac{\beta}{2}p^2\boldsymbol{\epsilon}^*\cdot\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)-2\mathrm{i}\pi p\boldsymbol{\epsilon}^*\cdot\boldsymbol{v}}\,\vartheta_{-K\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}-p\tfrac{\beta}{2}\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)\big|\tfrac{\beta}{2}\boldsymbol{\tau}\big), \tag{5.41}
$$

which holds for any $\boldsymbol{v} \in \mathbb{C}^g$. We multiply the outcome with the same factor for $p \to -p$ and obtain:

$$
\begin{aligned}
& \vartheta_{-(K+p)\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}\big|\tfrac{\beta}{2}\boldsymbol{\tau}\big)\vartheta_{-(K-p)\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}\big|\tfrac{\beta}{2}\boldsymbol{\tau}\big) \\
& = e^{\mathrm{i}\pi\beta p^2\boldsymbol{\epsilon}^*\cdot\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)}\vartheta_{-K\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}-\tfrac{\beta}{2}p\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)\big|\tfrac{\beta}{2}\boldsymbol{\tau}\big)\vartheta_{-K\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}+\tfrac{\beta}{2}p\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)\big|\tfrac{\beta}{2}\boldsymbol{\tau}\big).
\end{aligned}
$$

Inserting in (5.40) yields the claim. □

5.5 Derivation of the theta identities

We shall now state and prove the main theorems. For each case $\beta \in \{1, 2, 4\}$ we give both a proof based on the analysis in the previous sections, and a second, direct proof based on geometric arguments. We recall the definition of the meromorphic 1-form η on $\widetilde{C}$ that appeared in (5.37)

$$\eta(z) = \frac{E(\infty_+, \infty_-)}{E(z, \infty_+)E(z, \infty_-)}.$$

Throughout this section, the Abel map will always be based at ∞_+, and $\boldsymbol{u}(\infty_-)$ is computed using a path from ∞_+ to ∞_- which does not cross any of the representatives $(\mathcal{A}_h, \mathcal{B}_h)_{h=1}^g$ obtained by analytic continuation from the ones in Section 5.3.3, see the discussion above Lemma 5.3.14. Furthermore, we use the following notation: let $z_1, \ldots, z_k, z'_1, \ldots, z'_{k'} \in \tilde{C}$, we write

$$\boldsymbol{u}\left(z_1 + \cdots + z_l - z'_1 - \cdots - z_{k'}\right) := \sum_{i=1}^{k} \boldsymbol{u}(z_i) - \sum_{j=1}^{k'} \boldsymbol{u}(z'_j).$$

The formal sum in the left-hand side is commutative.

5.5.1 The $\beta = 2$ formula

Theorem 5.5.1. *Consider a hyperelliptic curve $\hat{C}$, and let $z, z', w, w' \in \widetilde{C}$, and $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{R}^g$. Then, we have:*

$$\begin{aligned}
&(X(w) - X(z'))(X(z) - X(w'))\frac{E(z,w)E(z',w')}{E(w,z')E(z,w')}\frac{\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(\boldsymbol{u}(z - z' + w - w')\big|\boldsymbol{\tau}\big)}{E(z,z')E(w,w')}\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}(\boldsymbol{0}|\boldsymbol{\tau}) \\
&- (X(z) - X(w))(X(z') - X(w'))\frac{\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}(\boldsymbol{u}(z-z')|\boldsymbol{\tau})}{E(z,z')}\frac{\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(\boldsymbol{u}(w-w')\big|\boldsymbol{\tau}\big)}{E(w,w')} \\
&= E(z,w)E(z',w')\left(\prod_{p\in\{z,z',w,w'\}}\eta(p)\right)\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(\boldsymbol{u}(z+w-\infty_-)\big|\boldsymbol{\tau}\big)\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(\boldsymbol{u}(-z'-w'+\infty_-)\big|\boldsymbol{\tau}\big).
\end{aligned} \tag{5.42}$$

Notice that each of the three terms is actually a $\frac{1}{2}$-form on the universal cover $\widetilde{C}$ in each of the variable z, z', w, w'. Even if $\boldsymbol{\mu} = 0$, it does not descend to the curve itself. As the proof will show, the formula holds equally well for arbitrary $\boldsymbol{\mu}, \boldsymbol{\nu}$ complex, in particular, if we shift all arguments of the theta functions by the same arbitrary but common $\boldsymbol{\nu} \in \mathbb{C}^g$.

Proof. Consider first the case where the Weierstraß points of $\hat{C}$ are real. By Proposition 5.3.11, we can find a β-ensemble whose spectral curve has $\hat{C}$ as underlying Riemann surface and for which the results of Section 5.4 apply (by construction the potential is off-critical). We then express the identity of Theorem 5.2.6 for $m_1 = m_2 = 1$, for $x = X(z)$, $x' = X(z')$, $\tilde{x} = X(w)$ and $\tilde{x}' = X(w')$ pairwise distinct points in $\mathbb{C} \setminus A$, that determine unique points $z, z', w, w' \in C_+$. Taking into account the definition of $\mathcal{K}$ in (5.39), we get:

$$\begin{aligned}
&\left\langle \frac{\det(x-\Lambda)\det(\tilde{x}-\Lambda)}{\det(x'-\Lambda)\det(\tilde{x}'-\Lambda)} \right\rangle_N^V \\
&= (x-x')(\tilde{x}-\tilde{x}')\frac{N}{N+1}\frac{Z^V_{N-1}Z^V_{N+1}}{(Z^V_N)^2}\mathcal{K}_{N-1}^{\frac{N}{N-1}V}\left(\begin{smallmatrix}1 & 1\\ x & \tilde{x}\end{smallmatrix}\right)\mathcal{K}_{N+1}^{\frac{N}{N+1}V}\left(\begin{smallmatrix}-1 & -1\\ \tilde{x}' & x'\end{smallmatrix}\right) + \mathcal{K}_N^V\left(\begin{smallmatrix}1 & -1\\ x & x'\end{smallmatrix}\right)\mathcal{K}_N^V\left(\begin{smallmatrix}1 & -1\\ \tilde{x} & \tilde{x}'\end{smallmatrix}\right).
\end{aligned} \tag{5.43}$$

Let us first consider the asymptotics of left-hand side as $N \to \infty$. We have $\boldsymbol{v}_{\mathrm{eq}} = 0$ since $\beta = 2$.

Coming back to Theorem 5.4.2 and using Lemma 5.4.7 for the $\mathcal{Q}$-terms involving two different variables:

$$\begin{aligned}
&\left\langle \frac{\det(x-\Lambda)\det(\tilde{x}-\Lambda)}{\det(x'-\Lambda)\det(\tilde{x}'-\Lambda)} \right\rangle_N^V \\
&\sim e^{N\mathcal{L}\left[\ln\left(\frac{(x-\bullet)(\tilde{x}-\bullet)}{(x'-\bullet)(\tilde{x}'-\bullet)}\right)\right]+\mathcal{H}\left[\ln\left(\frac{(x-\bullet)(\tilde{x}-\bullet)}{(x'-\bullet)(\tilde{x}'-\bullet)}\right)\right]+\frac{1}{2}\sum_{\xi\in\{x,\tilde{x},x',\tilde{x}'\}}\mathcal{Q}[\ln(\xi-\bullet),\ln(\xi-\bullet)]} \\
&\times \frac{E(z,w)E(z',w')E(z,\infty_+)E(z',\infty_+)E(w,\infty_+)E(w',\infty_+)\big(\mathrm{d}\zeta_{\infty_+}(\infty_+)\big)^2}{E(z,z')E(z,w')E(w,z')E(w,w')} \\
&\times \frac{(x-x')(x-\tilde{x}')(\tilde{x}-x')(\tilde{x}-\tilde{x}')}{(x-\tilde{x})(x'-\tilde{x}')} \\
&\times \frac{\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{u}(z)-\boldsymbol{u}(z')+\boldsymbol{u}(w)-\boldsymbol{u}(w')\big|\boldsymbol{\tau}\big)}{\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\mathbf{0}\big|\boldsymbol{\tau}\big)}.
\end{aligned} \tag{5.44}$$

For the asymptotics of the first term of the right-hand side of (5.43), we use the $\beta = 2$ specialisation of Lemma 5.4.9 with $p = 1$ for the ratio of partition functions, and Lemma 5.4.8 for the $\mathcal{K}$-factors with $K = N$, $M = N \mp 1$ and $(c,\tilde{c}) = (\pm 1, \pm 1)$, that is $p = \mp 1$. The outcome is

$$\begin{aligned}
&(x-x')(\tilde{x}-\tilde{x}')\frac{N}{N+1}\frac{Z_{N-1}^V Z_{N+1}^V}{(Z_N^V)^2}\mathcal{K}_{N-1}^{\frac{N}{N-1}V}\big(\begin{smallmatrix}1 & 1\\ x & \tilde{x}\end{smallmatrix}\big)\mathcal{K}_{N+1}^{\frac{N}{N+1}V}\big(\begin{smallmatrix}-1 & -1\\ \tilde{x}' & x'\end{smallmatrix}\big) \\
&\sim e^{2\mathcal{E}[\mu_{\mathrm{eq}}]+2\mathcal{L}[V]+\mathcal{Q}[V,V]-2\mathrm{i}\pi\boldsymbol{\epsilon}^*\cdot(\boldsymbol{u}(z)+\boldsymbol{u}(z')+\boldsymbol{u}(w)+\boldsymbol{u}(w'))} \\
&\times e^{N\mathcal{L}\left[\ln\left(\frac{(x-\bullet)(\tilde{x}-\bullet)}{(x'-\bullet)(\tilde{x}'-\bullet)}\right)\right]+\mathcal{H}\left[\ln\left(\frac{(x-\bullet)(\tilde{x}-\bullet)}{(x'-\bullet)(\tilde{x}'-\bullet)}\right)\right]+\frac{1}{2}\sum_{\xi\in\{x,x',\tilde{x},\tilde{x}'\}}\mathcal{Q}[\ln(\xi-\bullet),\ln(\xi-\bullet)]} \\
&\times \frac{E(z,w)E(z',w')(x-x')(\tilde{x}-\tilde{x}')}{(x-\tilde{x})(x'-\tilde{x}')}\prod_{p\in\{z,z',w,w'\}}\eta(p)E(p,\infty_+)\sqrt{\mathrm{d}\zeta_{\infty_+}(\infty_+)} \\
&\times \frac{\vartheta_{-(N-1)\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{u}(z+w-\infty_-)-\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)\big|\boldsymbol{\tau}\big)\vartheta_{-(N+1)\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{u}(-z'-w'+\infty_-)+\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)\big|\boldsymbol{\tau}\big)}{\vartheta^2_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\mathbf{0}\big|\boldsymbol{\tau}\big)}.
\end{aligned} \tag{5.45}$$

In the last line we can restore a characteristic $-N\boldsymbol{\epsilon}^*$ in the theta functions thanks to (5.41), which we have to use respectively with $\boldsymbol{v} = \boldsymbol{u}(z)+\boldsymbol{u}(w)-\boldsymbol{u}(\infty_-)-\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)$ and $\boldsymbol{v} = -\boldsymbol{u}(z')-\boldsymbol{u}(w')+\boldsymbol{u}(\infty_-)+\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)$. The outcome is

$$\begin{aligned}
&\vartheta_{-(N-1)\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{u}(z)+\boldsymbol{u}(w)-\boldsymbol{u}(\infty_-)-\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)\big|\boldsymbol{\tau}\big)\vartheta_{-(N+1)\boldsymbol{\epsilon}^*,\mathbf{0}}\big(-\boldsymbol{u}(z')-\boldsymbol{u}(w')+\ \boldsymbol{u}(\infty_-)+\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)\big|\boldsymbol{\tau}\big) \\
&= e^{2\mathrm{i}\pi\boldsymbol{\epsilon}^*\cdot\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)+2\mathrm{i}\pi\boldsymbol{\epsilon}^*\cdot(\boldsymbol{u}(z)+\boldsymbol{u}(z')+\boldsymbol{u}(w)+\boldsymbol{u}(w')-2\boldsymbol{u}(\infty_-)-2\boldsymbol{\tau}(\boldsymbol{\epsilon}^*))} \\
&\times \vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{u}(z)+\boldsymbol{u}(w)-\boldsymbol{u}(\infty_-)\big|\boldsymbol{\tau}\big)\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(-\boldsymbol{u}(z')-\boldsymbol{u}(w')+\boldsymbol{u}(\infty_-)\big|\boldsymbol{\tau}\big).
\end{aligned}$$

Inserting this in (5.45) gives an exponential prefactor

$$e^{2\mathcal{E}[\mu_{\mathrm{eq}}]+2\mathcal{L}[V]+\mathcal{Q}[V,V]+4\mathrm{i}\pi\boldsymbol{\epsilon}^*\cdot(\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)+\boldsymbol{u}(\infty_-))}$$

which would be equal to 1 if we had Proposition 5.4.3. We will establish Proposition 5.4.3 as a byproduct in Section 5.5.4; for the moment, we proceed assuming it holds. We get:

$$\begin{aligned}
&(x-x')(\tilde{x}-\tilde{x}')\frac{N}{N+1}\frac{Z_{N-1}^V Z_{N+1}^V}{(Z_N^V)^2}\mathcal{K}_{N-1}^{\frac{N}{N-1}V}\big(\begin{smallmatrix}1 & 1\\ x & \tilde{x}\end{smallmatrix}\big)\mathcal{K}_{N+1}^{\frac{N}{N+1}V}\big(\begin{smallmatrix}-1 & -1\\ \tilde{x}' & x'\end{smallmatrix}\big) \\
&\sim e^{N\mathcal{L}\left[\ln\left(\frac{(x-\bullet)(\tilde{x}-\bullet)}{(x'-\bullet)(\tilde{x}'-\bullet)}\right)\right]+\mathcal{H}\left[\ln\left(\frac{(x-\bullet)(\tilde{x}-\bullet)}{(x'-\bullet)(\tilde{x}'-\bullet)}\right)\right]+\frac{1}{2}\sum_{\xi\in\{x,x',\tilde{x},\tilde{x}'\}}\mathcal{Q}[\ln(\xi-\bullet),\ln(\xi-\bullet)]} \\
&\times \frac{E(z,w)E(z',w')(x-x')(\tilde{x}-\tilde{x}')}{(x-\tilde{x})(x'-\tilde{x}')}\prod_{p\in\{z,z',w,w'\}}\eta(p)E(p,\infty_+)\sqrt{\mathrm{d}\zeta_{\infty_+}(\infty_+)} \\
&\times \frac{\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{u}(z)+\boldsymbol{u}(w)-\boldsymbol{u}(\infty_-)\big|\boldsymbol{\tau}\big)\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(-\boldsymbol{u}(z')-\boldsymbol{u}(w')+\boldsymbol{u}(\infty_+)\big|\boldsymbol{\tau}\big)}{\vartheta^2_{-N\boldsymbol{\epsilon}^*}\big(\mathbf{0}\big|\boldsymbol{\tau}\big)}.
\end{aligned} \tag{5.46}$$

The asymptotics of the second term is simpler as we just need to use the $p = 0$ case of Lemma 5.4.8. The outcome is:

$$\begin{aligned}\mathcal{K}_N^V\big(\begin{smallmatrix}1 & -1\\ x & x'\end{smallmatrix}\big)\mathcal{K}_N^V\big(\begin{smallmatrix}1 & -1\\ \tilde{x} & \tilde{x}'\end{smallmatrix}\big) &\sim e^{N\mathcal{L}\left[\ln\left(\frac{(x-\bullet)(\tilde{x}-\bullet)}{(x'-\bullet)(\tilde{x}'-\bullet)}\right)\right]+\mathcal{H}\left[\ln\left(\frac{(x-\bullet)(\tilde{x}-\bullet)}{(x'-\bullet)(\tilde{x}'-\bullet)}\right)\right]+\frac{1}{2}\sum_{\xi\in\{x,x',\tilde{x},\tilde{x}'\}}\mathcal{Q}[\ln(\xi-\bullet),\ln(\xi-\bullet)]} \\ &\quad\times\frac{(x-x')(\tilde{x}-\tilde{x}')}{E(z,z')E(w,w')}\prod_{p\in\{z,z',w,w'\}}E(p,\infty_+)\sqrt{\mathrm{d}\zeta_{\infty_+}(\infty_+)} \\ &\quad\times\frac{\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{u}(z)-\boldsymbol{u}(z')\big|\boldsymbol{\tau}\big)\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{u}(w)-\boldsymbol{u}(w')\big|\boldsymbol{\tau}\big)}{\vartheta^2_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\mathbf{0}\big|\boldsymbol{\tau}\big)}.\end{aligned} \tag{5.47}$$

We also observe that (5.44), (5.46) and (5.47) all have the same exponential factor involving $\mathcal{L},\mathcal{H}$ and $\mathcal{Q}$, the same factor

$$\prod_{p\in\{z,z',w,w'\}}E(p,\infty_+)\sqrt{\mathrm{d}\zeta_{\infty_+}(\infty_+)},$$

and the same squared ϑ in the denominator — except for (5.44) where the latter is not squared. After we cancel those and multiply further by

$$\frac{(x-\tilde{x})(x'-\tilde{x}')}{(x-x')(\tilde{x}-\tilde{x}')},$$

the identity (5.43) as $N\to\infty$ then becomes:

$$\begin{aligned}&\frac{E(z,w)E(z',w')\,(x-\tilde{x}')(\tilde{x}-x')}{E(z,z')E(z,w')E(w,z')E(w,w')}\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{u}(z-z'+w-w')\big|\boldsymbol{\tau}\big)\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\mathbf{0}\big|\boldsymbol{\tau}\big) \\ &= E(z,w)E(z',w')\eta(z)\eta(z')\eta(w)\eta(w')\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{u}(z+w-\infty_-)\big|\boldsymbol{\tau}\big)\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{u}(-z'-w'+\infty_-)\big|\boldsymbol{\tau}\big) \\ &\quad+\frac{(x-\tilde{x})(x'-\tilde{x}')}{E(z,z')E(w,w')}\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{u}(z-z')\big|\boldsymbol{\tau}\big)\vartheta_{-N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{u}(w-w')\big|\boldsymbol{\tau}\big)+o(1).\end{aligned} \tag{5.48}$$

Let $\boldsymbol{\mu}\in\mathbb{R}^g$. Let us assume temporarily that $\epsilon_1^*,\ldots,\epsilon_g^*$ are $\mathbb{Q}$-linearly independent. Then, by Kronecker's theorem, one can find an increasing sequence $(N^{(n)})_{n\geq 1}$ such that

$$\lim_{n\to\infty}\big(N^{(n)}\boldsymbol{\epsilon}^*+\lfloor -N^{(n)}\boldsymbol{\epsilon}^*\rfloor\big)=-\boldsymbol{\mu},$$

where the integer part is applied to each component of the vector. Using $N=N^{(n)}$ in (5.48) and letting $n\to\infty$ we get the same identity without the $o(1)$ and with characteristic $\boldsymbol{\mu},\mathbf{0}$ for all theta functions, which is (5.42) with $\boldsymbol{\nu}=\mathbf{0}$ and pairwise distinct points $z,z',w,w'\in C_+$.

If $\epsilon_1^*,\ldots,\epsilon_g^*$ are not $\mathbb{Q}$-linearly independent, thanks to Corollary 5.3.13 we can take a sequence of β-ensembles whose spectral curve admits as underlying Riemann surface a hyperelliptic curve $\hat{C}^{(n)}$ with real Weierstraß points converging to those of $\hat{C}$, and whose filling fractions are $\mathbb{Q}$-linearly independent. By the previous argument, we know (5.42) with arbitrary $\boldsymbol{\mu}\in\mathbb{R}^g$ and $\boldsymbol{\nu}=\mathbf{0}$ and $z,z',w,w'\in C_+^{(n)}$ pairwise distinct. Since all the members of this identity are continuous in the real Weierstraß points $a_0<b_0<\cdots<a_g<b_g$ while the values $X(z),X(z'),X(w),X(w')\in\mathbb{C}$ are fixed away from them, taking $n\to\infty$ shows that the formula also holds for $\hat{C}$, $\boldsymbol{\nu}=0$ and pairwise distinct points $z,z',w,w'\in C_+$. Let us call $(\star)$ this formula. From there, we can derive the desired identity in full generality by using repeatedly analytic continuation, as follows.

Firstly, all terms in $(\star)$ are holomorphic functions of $\boldsymbol{\mu}\in\mathbb{C}^g$, and the identity holds for real $\boldsymbol{\mu}$. Therefore, it must hold as well for complex $\boldsymbol{\mu}$. In particular, we can replace $\boldsymbol{\mu}$ with $\boldsymbol{\mu}+\boldsymbol{\tau}^{-1}(\boldsymbol{\nu})$ for arbitrary $\boldsymbol{\mu},\boldsymbol{\nu}\in\mathbb{R}^g$, and rewrite all theta functions as

$$\vartheta_{\boldsymbol{\mu}+\boldsymbol{\tau}(\boldsymbol{\nu}),\mathbf{0}}(\boldsymbol{z}|\boldsymbol{\tau})=e^{\mathrm{i}\pi\boldsymbol{\tau}^{-1}(\boldsymbol{\nu})\cdot\boldsymbol{\nu}+2\mathrm{i}\pi\boldsymbol{\tau}^{-1}(\boldsymbol{\nu})\cdot\boldsymbol{z}}\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}(\boldsymbol{z}|\boldsymbol{\tau}).$$

The resulting phase is common to the three terms of the identity, therefore ($\star$) is valid for arbitrary $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{C}^g$, and a fortiori for arbitrary real $\boldsymbol{\mu}, \boldsymbol{\nu}$.

Secondly, fix $R > 0$ large enough, and let $\widetilde{\boldsymbol{\Delta}}_{2g+2}(R)$ be the subset of points in our parameter space of marked hyperelliptic curves $\widetilde{\boldsymbol{\Delta}}_{2g+2}$ such that the X-image of the Weierstraß points have moduli $\leq R$. Having fixed and pairwise distinct values $x, x', \tilde{x}, \tilde{x}' \in \mathbb{C}$ such that $\max\big(|x|, |x'|, |\tilde{x}|, |\tilde{x}'|\big) \geq 2R$ determines a unique quadruple of analytic sections $z, z', w, w' : \widetilde{\boldsymbol{\Delta}}_{2g+2}(R) \to \hat{\boldsymbol{C}}$ such that $x = X(z)$, $x' = X(z')$, $\tilde{x} = X(w)$ and $\tilde{x}' = X(w')$. These sections represent points in the (moving with parameters) hyperelliptic curve. Due to Lemma 5.3.14 and the discussion preceding it, all terms in ($\star$) are holomorphic functions on $\widetilde{\boldsymbol{\Delta}}_{2g+2}(R)$, but since we know that ($\star$) holds in the connected component of the base point in the real locus of $\widetilde{\boldsymbol{\Delta}}_{2g+2}(R)$, it must also hold over the whole $\widetilde{\boldsymbol{\Delta}}_{2g+2}(R)$ with $\max\big(|x|, |x'|, |\tilde{x}|, |\tilde{x}'|\big) \geq 2R$. Now rather fixing a marked hyperelliptic curve corresponding to a point in $\widetilde{\boldsymbol{\Delta}}_{2g+2}(R)$, the identity ($\star$) is valid for points z, z', w, w' in a neighborhood of ∞_+, but since it can be seen as an identity involving only meromorphic functions of $z, z', w, w' \in \widetilde{C}$, it must hold for arbitrary quadruple of points in $\widetilde{C}$. Eventually as R is arbitrary in the argument we get (5.42) over the whole parameter space $\widetilde{\boldsymbol{\Delta}}_{2g+2}$ and quadruples of points in the universal cover of the associated hyperelliptic curve. □

This formula implies the Fay identity, as we now show.

Proposition 5.5.2. *Formula* (5.42) *implies the Fay identity* (2.26) *for hyperelliptic curves.*

Proof. The key remark is that z and w play almost symmetric roles in (5.42). We write the two equations obtained when exchanging z and w, specialised to $\boldsymbol{\mu} = 0$ and $\boldsymbol{\nu} \in \mathbb{C}^g$ arbitrary but rather transferred to the argument of the theta functions, so that everything is expressed in terms of $\theta = \vartheta_{\mathbf{0},\mathbf{0}}$:

$$\begin{aligned}
&(X(w) - X(z'))(X(z) - X(w'))\frac{E(z,w)E(z',w')}{E(w,z')E(z,w')}\frac{\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(z - z' + w - w')\big|\boldsymbol{\tau}\big)}{E(z,z')E(w,w')}\theta\big(\boldsymbol{\nu}\big|\boldsymbol{\tau}\big) \\
&\quad - (X(z) - X(w))(X(z') - X(w'))\frac{\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(z - z')\big|\boldsymbol{\tau}\big)}{E(z,z')}\frac{\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(w - w')\big|\boldsymbol{\tau}\big)}{E(w,w')} \\
&= E(z,w)E(z',w')\eta(z)\eta(z')\eta(w)\eta(w')\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(z + w - \infty_-)\big|\boldsymbol{\tau}\big)\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(-z' - w' + \infty_-)\big|\boldsymbol{\tau}\big) \\
&= (X(z) - X(z'))(X(w) - X(w'))\frac{E(z,w)E(z',w')}{E(z,z')E(w,w')}\frac{\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(z - z' + w - w')\big|\boldsymbol{\tau}\big)}{E(w,z')E(z,w')}\theta\big(\boldsymbol{\nu}\big|\boldsymbol{\tau}\big) \\
&\quad - (X(z) - X(w))(X(z') - X(w'))\frac{\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(w - z')\big|\boldsymbol{\tau}\big)}{E(w,z')}\frac{\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(z - w')\big|\boldsymbol{\tau}\big)}{E(z,w')}\,.
\end{aligned} \tag{5.49}$$

Subtracting the first member from the third member of the equalities, grouping the terms together and dividing by $(X(z) - X(w))(X(z') - X(w'))$ yields the identity

$$\begin{aligned}
0 &= \frac{E(z,w)E(z',w')}{E(w,z')E(z,w')E(z,z')E(w,w')}\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(z) - \boldsymbol{u}(z') + \boldsymbol{u}(w) - \boldsymbol{u}(w')\big|\boldsymbol{\tau}\big)\theta\big(\boldsymbol{\nu}\big|\boldsymbol{\tau}\big) \\
&\quad - \frac{\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(w) - \boldsymbol{u}(z')\big|\boldsymbol{\tau}\big)\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(z) - \boldsymbol{u}(w')\big|\boldsymbol{\tau}\big)}{E(w,z')E(z,w')} \\
&\quad + \frac{\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(z) - \boldsymbol{u}(z')\big|\boldsymbol{\tau}\big)\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(w) - \boldsymbol{u}(w')\big|\boldsymbol{\tau}\big)}{E(z,z')E(w,w')},
\end{aligned}$$

which is exactly the Fay identity (2.26) after we replace the prime form with its expression (5.18) and take $(z_1, z_2, z_3, z_4) = (z, w, z', w')$. □

Finally, we provide a direct proof of (5.42) based on complex analysis. This proof is based on a classical theorem of Riemann which we recall for the convenience of the reader.

Theorem 5.5.3 ([Mum83, Theorem 3.1])**.** *There is a vector* $\boldsymbol{k} \in \mathbb{C}^g$ *such that for all* $\nu' \in \mathbb{C}^g$*, the function* $z \mapsto \theta\big(\boldsymbol{\nu}' + \boldsymbol{u}(z)\big|\boldsymbol{\tau}\big)$ *of* $z \in \widetilde{C}$ *either vanishes identically or has* g *zeroes* $w_1, \ldots, w_g$ *in a fundamental domain, satisfying*

$$\sum_{h=1}^{g} \boldsymbol{u}(w_h) = -\boldsymbol{\nu} + \boldsymbol{k} \mod \mathbb{L}.$$

$\boldsymbol{k}$ is called vector of Riemann constants.

Direct geometric proof of Theorem 5.5.1. It suffices to prove the identity for $\boldsymbol{\mu} = 0$, since we still have $\boldsymbol{\nu} \in \mathbb{C}^g$ arbitrary which allow reconstructing arbitrary characteristics. Let $\boldsymbol{\nu}' \in \mathbb{C}^g$. Riemann's theorem 5.5.3 implies that seen as a function of $z \in \widetilde{C}$, the theta function $z \mapsto \theta\big(\boldsymbol{\nu}' + \boldsymbol{u}(z)\big|\boldsymbol{\tau}\big)$ is either identically zero of has g zeroes in a fundamental domain. Let $\mathcal{D}_{\boldsymbol{\nu}'}$ be its zero divisor. We apply this to $\boldsymbol{\nu}' = \boldsymbol{\nu} + \boldsymbol{u}(w) - \boldsymbol{u}(z') - \boldsymbol{u}(w')$ with $\boldsymbol{\nu}, z', w, w'$ generic such that it is not in the theta divisor. A classical consequence of Riemann's theorem [FK92, Theorem VI.3.3.] is that meromorphic functions on C with pole divisor at most $\mathcal{D}_{\boldsymbol{\nu}'}$ are constant. For convenience, write

$$\begin{aligned}
\mathfrak{c}_1 &= \frac{(X(w) - X(z'))(X(z) - X(w'))E(z,w)E(z',w')}{E(w,z')E(z,w')E(z,z')E(w,w')} \\
\mathfrak{c}_2 &= -\frac{(X(z) - X(w))(X(z') - X(w'))}{E(z,z')E(w,w')} \\
\mathfrak{c}_3 &= E(z,w)E(z',w')\eta(z)\eta(z')\eta(w)\eta(w')
\end{aligned}$$

and consider

$$\begin{aligned}
\Psi(z) = \frac{\mathfrak{c}_2}{\mathfrak{c}_1}\, & \frac{\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(z) - \boldsymbol{u}(z')\big|\boldsymbol{\tau}\big)\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(w) - \boldsymbol{u}(w')\big|\boldsymbol{\tau}\big)}{\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(z) - \boldsymbol{u}(z') + \boldsymbol{u}(w) - \boldsymbol{u}(w')\big|\boldsymbol{\tau}\big)\theta\big(\boldsymbol{\nu}\big|\boldsymbol{\tau}\big)} \\
& + \frac{\mathfrak{c}_3}{\mathfrak{c}_1}\, \frac{\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(z) + \boldsymbol{u}(w) - \boldsymbol{u}(\infty_-)\big|\boldsymbol{\tau}\big)\theta\big(\boldsymbol{\nu} - \boldsymbol{u}(z') - \boldsymbol{u}(w') + \boldsymbol{u}(\infty_-)\big|\boldsymbol{\tau}\big)}{\theta\big(\boldsymbol{\nu} + \boldsymbol{u}(z) - \boldsymbol{u}(z') + \boldsymbol{u}(w) - \boldsymbol{u}(w')\big|\boldsymbol{\tau}\big)\theta\big(\boldsymbol{\nu}\big|\boldsymbol{\tau}\big)}.
\end{aligned}$$

This is a meromorphic function of $z \in \hat{C}$. We have seen that the theta function in the denominator has g zeroes – which are thus poles of Ψ. We now consider the poles of the other factors: the zeroes of $\mathfrak{c}_1$ and the poles of $\mathfrak{c}_2$ and $\mathfrak{c}_3$. The coefficient $\mathfrak{c}_1$ has a simple zero at $z = \jmath(w')$, where $\jmath$ is the hyperelliptic involution, and at $z = w$. The coefficients $\mathfrak{c}_2$ and $\mathfrak{c}_3$ have simple poles at $z = z'$. Accordingly, both ratios $\frac{\mathfrak{c}_2}{\mathfrak{c}_1}$ and $\frac{\mathfrak{c}_3}{\mathfrak{c}_1}$ have a simple pole only at $z = \jmath(w')$. A careful computation of the residues taking into account $\boldsymbol{u}(\jmath(w')) = \boldsymbol{u}(\infty_-) - \boldsymbol{u}(w')$ shows that Ψ does not have a pole when $z = \jmath(w')$. Notice that there are no pole at $\infty_\pm$ as poles coming from linear terms are cancelled by other linear terms or the form η. We conclude that the divisor of poles of Ψ is at most $\mathcal{D}_{\boldsymbol{\nu}'}$. Thus, it is a constant function of z. A similar argument with the other variables would show that it is a constant function of z, z', w, w'. By sending the points z', w, w' to z one after the other, and we arrive to $\Psi = 1$. □

5.5.2 The $\beta = 1$ formula

Theorem 5.5.4. *Consider a marked hyperelliptic curve $\hat{C}$ and let $z_1, z_1', z_2, z_2' \in \widetilde{C}$ and $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{R}^g$. Writing $x_i = X(z_i)$ and $x_i' = X(z_i')$, we have*

$$\begin{aligned}
&\left(\frac{E(z_1,z_2)E(z_1',z_2')}{E(z_1,z_1')E(z_1,z_2')E(z_2,z_1')E(z_2,z_2')}\right)^2 \vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(\boldsymbol{u}(z_1)-\boldsymbol{u}(z_1')+\boldsymbol{u}(z_2)-\boldsymbol{u}(z_2')\big|\tfrac{\tau}{2}\big)\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(\boldsymbol{0}\big|\tfrac{\tau}{2}\big) \\
&= \frac{(x_1-x_2)(x_1'-x_2')}{(x_1-x_1')(x_2-x_2')}\,\frac{\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(\boldsymbol{u}(z_1)-\boldsymbol{u}(z_2')\big|\tfrac{\tau}{2}\big)}{E(z_1,z_2')^2}\,\frac{\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(\boldsymbol{u}(z_2)-\boldsymbol{u}(z_1')\big|\tfrac{\tau}{2}\big)}{E(z_1',z_2)^2} \\
&\quad - \frac{(x_1-x_2)(x_1'-x_2')}{(x_1-x_2')(x_2-x_1')}\,\frac{\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(\boldsymbol{u}(z_1)-\boldsymbol{u}(z_1')\big|\tfrac{\tau}{2}\big)}{E(z_1,z_1')^2}\,\frac{\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(\boldsymbol{u}(z_2)-\boldsymbol{u}(z_2')\big|\tfrac{\tau}{2}\big)}{E(z_2',z_2)^2} \\
&\quad + \frac{\big(E(z_1,z_2)E(z_1',z_2')\eta(z_1)\eta(z_1')\eta(z_2)\eta(z_2')\big)^2}{(x_1-x_1')(x_1-x_2')(x_2-x_1')(x_2-x_2')} \\
&\quad \times \vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(\boldsymbol{u}(z_1)+\boldsymbol{u}(z_2)-\boldsymbol{u}(\infty_-)\big|\tfrac{\tau}{2}\big)\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(-\boldsymbol{u}(z_1')-\boldsymbol{u}(z_2')+\boldsymbol{u}(\infty_-)\big|\tfrac{\tau}{2}\big).
\end{aligned} \tag{5.50}$$

Proof. The strategy is similar to the proof for $\beta = 2$ in Theorem 5.5.1. In particular, we first prove an asymptotic identity for hyperelliptic curves arising from $\beta = 1$ ensembles, use approximations to get arbitrary characteristic $\boldsymbol{\mu}, \boldsymbol{0}$, and then analytic continuation to get the identity for marked hyperelliptic curves with arbitrary complex Weierstraß points and characteristics $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{R}^g$.

The starting point is the exact identity of Theorem 5.2.7 in the simplest non-trivial case, i.e. $m = 2$ (pfaffian of size 4). Taking $x_1, x_1', x_2, x_2' \in \mathbb{C} \setminus A$ pairwise distinct, this gives

$$\begin{aligned}
&\left\langle \frac{\det(x_1-\Lambda)\det(x_2-\Lambda)}{\det(x_1'-\Lambda)\det(x_2'-\Lambda)} \right\rangle_{2N}^{V} \\
&= \frac{2N(2N-1)}{(2N+2)(2N+1)}(x_1-x_1')(x_2-x_2')(x_1-x_2')(x_2-x_1')\frac{Z_{2N-2}^{V}Z_{2N+2}^{V}}{(Z_{2N}^{V})^2} \\
&\qquad \times \mathcal{K}_{2N-2}^{\frac{2N}{2N-2}V}\left(\begin{smallmatrix} 1 & 1 \\ x_1 & x_2 \end{smallmatrix}\right)\mathcal{K}_{2N+2}^{\frac{2N}{2N+2}V}\left(\begin{smallmatrix} -1 & -1 \\ x_1' & x_2' \end{smallmatrix}\right) \\
&\quad - \frac{(x_1-x_2')(x_2-x_1')}{(x_1-x_2)(x_1'-x_2')}\mathcal{K}_{2N}^{V}\left(\begin{smallmatrix} 1 & -1 \\ x_1 & x_1' \end{smallmatrix}\right)\mathcal{K}_{2N}^{V}\left(\begin{smallmatrix} 1 & -1 \\ x_2 & x_2' \end{smallmatrix}\right) + \frac{(x_1-x_1')(x_2-x_2')}{(x_1-x_2)(x_1'-x_2')}\mathcal{K}_{2N}^{V}\left(\begin{smallmatrix} 1 & -1 \\ x_1 & x_2' \end{smallmatrix}\right)\mathcal{K}_{2N}^{V}\left(\begin{smallmatrix} 1 & -1 \\ x_2 & x_1' \end{smallmatrix}\right).
\end{aligned} \tag{5.51}$$

As the computation is similar to $\beta = 2$ we only streamline it. Let $z_1, z_1', z_2, z_2' \in C_+$ be such that $x_i = X(z_i)$ and $x_i' = X(z_i')$. The asymptotic equivalent of the left-hand side of (5.51) as $N \to \infty$ is obtained from Theorem 5.4.2:

$$\begin{aligned}
&e^{(2N\mathcal{L}+\mathcal{H})\left[\ln\left(\frac{(x_1-\bullet)(x_2-\bullet)}{(x_1'-\bullet)(x_2'-\bullet)}\right)\right]+\frac{1}{2}\sum_{\xi\in\{x_1,x_1',x_2,x_2'\}}\mathcal{Q}[\ln(\xi-\bullet),\ln(\xi-\bullet)]} \\
&\times \left(\prod_{p\in\{z_1,z_1',z_2,z_2'\}} E(p,\infty_+)\sqrt{\mathrm{d}\zeta_{\infty_+}(\infty_+)}\right)^2 \\
&\times \left(\frac{E(z_1,z_2)E(z_1',z_2')}{E(z_1,z_1')E(z_1,z_2')E(z_2,z_1')E(z_2,z_2')}\;\frac{(x_1-x_1')(x_1-x_2')(x_2-x_1')(x_2-x_2')}{(x_1-x_2)(x_1'-x_2')}\right)^2 \\
&\times \frac{\vartheta_{-2N\boldsymbol{\epsilon}^*,\boldsymbol{0}}\big(\boldsymbol{v}_{\mathrm{eq}}+\boldsymbol{u}(z_1)-\boldsymbol{u}(z_1')+\boldsymbol{u}(z_2)-\boldsymbol{u}(z_2')\big|\tfrac{\tau}{2}\big)}{\vartheta_{-2N\boldsymbol{\epsilon}^*,\boldsymbol{0}}\big(\boldsymbol{v}_{\mathrm{eq}}\big|\tfrac{\tau}{2}\big)}.
\end{aligned} \tag{5.52}$$

We have used Lemma 5.4.7 to evaluate the $\mathcal{Q}[\ln(\xi-\bullet),\ln(\xi'-\bullet)]$ terms that appear with $\xi \neq \xi'$. In the right-hand side of (5.51), we use the asymptotics of the 2-point kernel from Lemma 5.4.8 with $K = 2N$ and $p = \mp 2$.

Consider the asymptotics of the first term in the right-hand side of (5.51). It contains a product of theta functions with characteristic $-(2N \pm 2)\boldsymbol{\epsilon}^*$, which we can replace by two theta functions with same characteristic $-2N\boldsymbol{\epsilon}^*$ using (5.52) up to an extra exponential factor. The latter combines with the asymptotics of the ratio of partition functions of shifted size from Lemma 5.4.9 to reproduce a factor

$$e^{8\mathcal{E}[\mu_{\mathrm{eq}}]+4\mathcal{L}[V]+\mathcal{Q}[V,V]+8\mathrm{i}\pi\boldsymbol{\epsilon}^*\cdot(\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)+\boldsymbol{u}(\infty_-))}$$

which is equal to 1 due to Proposition 5.4.3, and to kill the factor of $e^{-4\mathrm{i}\pi\boldsymbol{\epsilon}^*\cdot(\boldsymbol{u}(z_1)+\boldsymbol{u}(z_2)+\boldsymbol{u}(z_1')+\boldsymbol{u}(z_2'))}$ coming from the use of (5.35). The other factors are the first line of (5.52) multiplied by

$$\begin{aligned}&\bigg(\frac{E(z_1,z_2)E(z_1',z_2')}{(x_1-x_2)(x_1'-x_2')}\prod_{p\in\{z_1,z_1',z_2,z_2'\}}\eta(p)E(p,\infty_+)\sqrt{\mathrm{d}\zeta_{\infty_+}(\infty_+)}\bigg)^2\\ &\times\vartheta_{-2N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}-\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)\big|\tfrac{\boldsymbol{\tau}}{2}\big)\vartheta_{-2N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}+\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)\big|\tfrac{\boldsymbol{\tau}}{2}\big)\\ &\times\vartheta_{-2N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}+\boldsymbol{u}(z_1)+\boldsymbol{u}(z_2)-\boldsymbol{u}(\infty_-)\big|\tfrac{\boldsymbol{\tau}}{2}\big)\vartheta_{-2N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}-\boldsymbol{u}(z_1')-\boldsymbol{u}(z_2')+\boldsymbol{u}(\infty_-)\big|\tfrac{\boldsymbol{\tau}}{2}\big),\end{aligned}$$

where the last line was already explained.

The asymptotics of the second and third terms in the right-hand side of (5.51) are more straightforward to get. They both contain the first line and the third line of (5.52), and the other asymptotic factors are

$$\begin{aligned}&-\frac{(x_1-x_2')(x_2-x_1')}{(x_1-x_2)(x_1'-x_2')}\bigg(\frac{(x_1-x_1')(x_2-x_2')}{E(z_1,z_1')E(z_2,z_2')}\prod_{p\in\{z_1,z_1',z_2,z_2'\}}E(p,\infty_+)\sqrt{\mathrm{d}\zeta_{\infty_+}(\infty_+)}\bigg)^2\\ &\times\vartheta_{-2N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}+\boldsymbol{u}(z_1)-\boldsymbol{u}(z_1')\big|\tfrac{\boldsymbol{\tau}}{2}\big)\vartheta_{-2N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}+\boldsymbol{u}(z_2)-\boldsymbol{u}(z_2')\big|\tfrac{\boldsymbol{\tau}}{2}\big)\end{aligned}$$

for the second term (including its sign), and

$$\begin{aligned}&\frac{(x_1-x_1')(x_2-x_2')}{(x_1-x_2)(x_1'-x_2')}\bigg(\frac{(x_1-x_2')(x_2-x_1')}{E(z_1,z_2')E(z_2,z_1')}\prod_{p\in\{z_1,z_1',z_2,z_2'\}}E(p,\infty_+)\sqrt{\mathrm{d}\zeta_{\infty_+}(\infty_+)}\bigg)^2\\ &\times\vartheta_{-2N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}+\boldsymbol{u}(z_1)-\boldsymbol{u}(z_2')\big|\tfrac{\boldsymbol{\tau}}{2}\big)\vartheta_{-2N\boldsymbol{\epsilon}^*,\mathbf{0}}\big(\boldsymbol{v}_{\mathrm{eq}}+\boldsymbol{u}(z_2)-\boldsymbol{u}(z_1')\big|\tfrac{\boldsymbol{\tau}}{2}\big).\end{aligned}$$

We then divide all terms by the first and second line of (5.52) (common factor to all terms) and by

$$\bigg(\frac{(x_1-x_1')(x_1-x_2')(x_2-x_1')(x_2-x_2')}{(x_1-x_2)(x_1'-x_2')}\bigg)^2$$

to arrive to (5.50) with $\boldsymbol{\mu}=-2N\boldsymbol{\epsilon}^*$, $\boldsymbol{\nu}=0$ and an extra $\boldsymbol{v}_{\mathrm{eq}}$ added to the arguments of all theta functions. Then, we repeat the end of the proof of Theorem 5.5.1 to get exactly and in full generality the claimed (5.5.1). □

Theorem 5.5.4 can be reformulated as an identity for theta functions with matrix of periods $\boldsymbol{\tau}$ instead of $\frac{\boldsymbol{\tau}}{2}$.

Lemma 5.5.5. *There is an equivalence between (5.50) for any $\boldsymbol{\mu},\boldsymbol{\nu}\in\mathbb{R}^g$, and the formula*

$$\begin{aligned}0=\;&\mathfrak{c}_1\,\vartheta_{\frac{\boldsymbol{\alpha}}{2},\mathbf{0}}\big(\boldsymbol{u}(z_1)-\boldsymbol{u}(z_1')+\boldsymbol{u}(z_2)-\boldsymbol{u}(z_2')\big|\boldsymbol{\tau}\big)\\ &+\mathfrak{c}_2\,\vartheta_{\frac{\boldsymbol{\alpha}}{2},\mathbf{0}}\big(\boldsymbol{u}(z_1)+\boldsymbol{u}(z_1')-\boldsymbol{u}(z_2)-\boldsymbol{u}(z_2')\big|\boldsymbol{\tau}\big)\\ &+\mathfrak{c}_3\,\vartheta_{\frac{\boldsymbol{\alpha}}{2},\mathbf{0}}\big(\boldsymbol{u}(z_1)-\boldsymbol{u}(z_1')-\boldsymbol{u}(z_2)+\boldsymbol{u}(z_2')\big|\boldsymbol{\tau}\big)\\ &+\mathfrak{c}_4\,\vartheta_{\frac{\boldsymbol{\alpha}}{2},\mathbf{0}}\big(\boldsymbol{u}(z_1)+\boldsymbol{u}(z_1')+\boldsymbol{u}(z_2)+\boldsymbol{u}(z_2')-2\boldsymbol{u}(\infty_-)\big|\boldsymbol{\tau}\big),\end{aligned}\qquad(5.53)$$

for any $\boldsymbol{\alpha} \in \mathbb{Z}^g/2\mathbb{Z}^g$, where

$$\mathfrak{c}_1 = -\left(\frac{E(z_1,z_2)E(z_1',z_2')}{E(z_1,z_1')E(z_1,z_2')E(z_2,z_1')E(z_2,z_2')}\right)^2,$$
$$\mathfrak{c}_2 = \frac{(x_1-x_2)(x_1'-x_2')}{(x_1-x_1')(x_2-x_2')}\frac{1}{\big(E(z_1,z_2')E(z_1',z_2)\big)^2},$$
$$\mathfrak{c}_3 = -\frac{(x_1-x_2)(x_1'-x_2')}{(x_1-x_2')(x_2-x_1')}\frac{1}{\big(E(z_1,z_1')E(z_2',z_2)\big)^2},$$
$$\mathfrak{c}_4 = \frac{\big(E(z_1,z_2)E(z_1',z_2')\eta(z_1)\eta(z_1')\eta(z_2)\eta(z_2')\big)^2}{(x_1-x_1')(x_1-x_2')(x_2-x_1')(x_2-x_2')}.$$

Proof. The trick is to use Riemann's binary addition theorem — see e.g. [Mum83, Equation 6.6]. It states that for any $\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\mu}',\boldsymbol{\nu}' \in \mathbb{R}^g$ and $\boldsymbol{z}_1,\boldsymbol{z}_2 \in \mathbb{C}^g$

$$\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(\boldsymbol{z}_1+\boldsymbol{z}_2\big|\tfrac{\boldsymbol{\tau}}{2}\big)\vartheta_{\boldsymbol{\mu}',\boldsymbol{\nu}'}\big(\boldsymbol{z}_1-\boldsymbol{z}_2\big|\tfrac{\boldsymbol{\tau}}{2}\big) = \sum_{\boldsymbol{\alpha}\in\mathbf{Z}^g/2\mathbf{Z}^g} \vartheta_{\frac{\boldsymbol{\mu}+\boldsymbol{\mu}'+\boldsymbol{\alpha}}{2},\boldsymbol{\nu}+\boldsymbol{\nu}'}\big(2\boldsymbol{z}_1\big|\boldsymbol{\tau}\big)\vartheta_{\frac{\boldsymbol{\mu}-\boldsymbol{\mu}'+\boldsymbol{\alpha}}{2},\boldsymbol{\nu}-\boldsymbol{\nu}'}\big(2\boldsymbol{z}_2\big|\boldsymbol{\tau}\big). \quad (5.54)$$

We apply the transformation (5.54) with $\boldsymbol{\mu}'=\boldsymbol{\mu}$ and $\boldsymbol{\nu}=\boldsymbol{\nu}'$ to each term in (5.50), writing it in the equivalent form

$$\sum_{\boldsymbol{\alpha}\in\mathbb{Z}^g/2\mathbb{Z}^g} \vartheta_{\boldsymbol{\mu}+\frac{\boldsymbol{\alpha}}{2},\boldsymbol{\nu}}\big(\boldsymbol{u}(z_1)-\boldsymbol{u}(z_1')+\boldsymbol{u}(z_2)-\boldsymbol{u}(z_2')\big|\boldsymbol{\tau}\big)\left(\sum_{i=1}^{4}\mathfrak{c}_i\,\vartheta_{\frac{\boldsymbol{\alpha}}{2},\boldsymbol{0}}\big(\boldsymbol{w}_i(z_1,z_1',z_2,z_2')\big|\boldsymbol{\tau}\big)\right), \quad (5.55)$$

where the $\boldsymbol{w}_i$ are exactly the four arguments of the theta functions appearing in (5.53). In this form, the converse implication is clear. The direct implication follows from the observation that $\boldsymbol{\nu}$ is arbitrary, and the family of functions $T_{\boldsymbol{\alpha}}(\boldsymbol{\nu}) = \vartheta_{\frac{\boldsymbol{\alpha}}{2},\boldsymbol{0}}\big(\boldsymbol{u}(z_1)-\boldsymbol{u}(z_1')+\boldsymbol{u}(z_2)-\boldsymbol{u}(z_2')\big|\boldsymbol{\tau}\big)$ indexed by $\boldsymbol{\alpha}\in\mathbb{Z}^g/2\mathbb{Z}^g$ are linearly independent, forcing the sum inside the bracket to vanish for each individual $\boldsymbol{\alpha}$. □

This is an identity involving only Riemann theta functions, for which we can offer a direct geometric proof, in a slightly more general form.

Theorem 5.5.6. *Equation 5.53 holds for any marked hyperelliptic curve for any characteristic $\boldsymbol{\mu},\boldsymbol{\nu}\in\mathbb{R}^g$ (instead of just half-integer characteristics).*

Direct geometric proof of Theorem 5.5.4. The strategy is similar of the direct proof of Theorem 5.5.1. We start without lack of generality to set $\boldsymbol{\mu}=0$ and for $\boldsymbol{\nu}\in\mathbb{C}^g$ arbitrary, consider

$$\begin{aligned}\Psi(z_1) = {}& \frac{\mathfrak{c}_2}{\mathfrak{c}_1}\frac{\theta\big(\boldsymbol{\nu}+\boldsymbol{u}(z_1)+\boldsymbol{u}(z_1')-\boldsymbol{u}(z_2)-\boldsymbol{u}(z_2')\big|\boldsymbol{\tau}\big)}{\theta\big(\boldsymbol{\nu}+\boldsymbol{u}(z_1)-\boldsymbol{u}(z_1')+\boldsymbol{u}(z_2)-\boldsymbol{u}(z_2')\big|\boldsymbol{\tau}\big)} \\ &+ \frac{\mathfrak{c}_3}{\mathfrak{c}_1}\frac{\theta\big(\boldsymbol{\nu}+\boldsymbol{u}(z_1)-\boldsymbol{u}(z_1')-\boldsymbol{u}(z_2)+\boldsymbol{u}(z_2')\big|\boldsymbol{\tau}\big)}{\theta\big(\boldsymbol{\nu}+\boldsymbol{u}(z_1)-\boldsymbol{u}(z_1')+\boldsymbol{u}(z_2)-\boldsymbol{u}(z_2')\big|\boldsymbol{\tau}\big)} \\ &+ \frac{\mathfrak{c}_4}{\mathfrak{c}_1}\frac{\theta\big(\boldsymbol{\nu}+\boldsymbol{u}(z_1)+\boldsymbol{u}(z_1')+\boldsymbol{u}(z_2)+\boldsymbol{u}(z_2')-2\boldsymbol{u}(\infty_-)\big|\boldsymbol{\tau}\big)}{\theta\big(\boldsymbol{\nu}+\boldsymbol{u}(z_1)-\boldsymbol{u}(z_1')+\boldsymbol{u}(z_2)-\boldsymbol{u}(z_2')\big|\boldsymbol{\tau}\big)}.\end{aligned}$$

This is a meromorphic function of $z_1\in\hat{C}$. We first analyse the poles that may come from the ratios of coefficients. The ratio $\frac{\mathfrak{c}_2}{\mathfrak{c}_1}$ has simple poles at $z_1=z_2$ and $z_1=\jmath(z_1')$, where $\jmath$ is the hyperelliptic involution. The ratio $\frac{\mathfrak{c}_3}{\mathfrak{c}_1}$ has simple poles only at $z_1=\jmath(z_2')$ and $z_1=z_2$. The ratio $\frac{\mathfrak{c}_4}{\mathfrak{c}_1}$ has simple poles only at $z_1=\jmath(z_2')$ and $z_1=\jmath(z_2')$. However, careful computation of the residues show that Ψ has none of these poles. Thus, the only poles of Ψ are the zeros of $z_1\mapsto\theta\big(\boldsymbol{\nu}+\boldsymbol{u}(z_1)-\boldsymbol{u}(z_1')+\boldsymbol{u}(z_2)-\boldsymbol{u}(z_2')\big|\boldsymbol{\tau}\big)$. As in the direct proof of Theorem 5.5.1, Riemann's Theorem implies that if we choose the points z_1',z_2,z_2' and the vector $\boldsymbol{\nu}$ generically, there are no nonconstant meromorphic function whose poles are the zeroes of this theta function. We deduce that $\Psi(z_1)$ does not depend on z_1. A similar argument shows that $\Psi(z_1)$ is independent of all points points z_1,z_1',z_2,z_2'. Sending z_1',z_2,z_2' successively to z_1 we find that the constant is 1. □

5.5.3 The $\beta = 4$ formula

The case $\beta = 4$ has the same structure as the $\beta = 1$ case of Theorem 5.5.4, except that the argument of the theta functions are doubled while we use the matrix $2\boldsymbol{\tau}$. This similarity is already manifest in the exact formulae of Theorems 5.2.7 and 5.2.8.

Theorem 5.5.7. *Consider a marked hyperelliptic curve $\hat{C}$, and let $z_1, z_1', z_2, z_2' \in \widetilde{C}$ and $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{C}^g$. Writing $x_i = X(z_i)$ and $x_i' = X(z_i')$, we have have:*

$$\begin{aligned}
&\bigg(\frac{E(z_1,z_2)E(z_1',z_2')}{E(z_1,z_1')E(z_1,z_2')E(z_2,z_1')E(z_2,z_2')}\bigg)^2 \vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(2(\boldsymbol{u}(z_1) - \boldsymbol{u}(z_1') + \boldsymbol{u}(z_2) - \boldsymbol{u}(z_2'))\big|2\boldsymbol{\tau}\big)\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(\boldsymbol{0}\big|2\boldsymbol{\tau}\big) \\
&- \frac{(x_1-x_2)(x_1'-x_2')}{(x_1-x_1')(x_2-x_2')}\,\frac{\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(2(\boldsymbol{u}(z_1)-\boldsymbol{u}(z_2'))\big|2\boldsymbol{\tau}\big)}{E(z_1,z_2')^2}\,\frac{\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(2(\boldsymbol{u}(z_2)-\boldsymbol{u}(z_1'))\big|2\tau\big)}{E(z_1',z_2)^2} \\
&+ \frac{(x_1-x_2)(x_1'-x_2')}{(x_1-x_2')(x_2-x_1')}\,\frac{\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(2(\boldsymbol{u}(z_1)-\boldsymbol{u}(z_1'))\big|2\boldsymbol{\tau}\big)}{E(z_1,z_1')^2}\,\frac{\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(2(\boldsymbol{u}(z_2)-\boldsymbol{u}(z_2'))\big|2\boldsymbol{\tau}\big)}{E(z_2',z_2)^2} \\
&= \frac{\big(E(z_1,z_2)E(z_1',z_2')\eta(z_1)\eta(z_2)\eta(z_1')\eta(z_2')\big)^2}{(x_1-x_1')(x_1-x_2')(x_2-x_1')(x_2-x_2')} \\
&\quad\times \vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(2(\boldsymbol{u}(z_1)+\boldsymbol{u}(z_2)-\boldsymbol{u}(\infty_-))\big|2\boldsymbol{\tau}\big)\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(2(-\boldsymbol{u}(z_1')-\boldsymbol{u}(z_2')+\boldsymbol{u}(\infty_-))\big|2\boldsymbol{\tau}\big).
\end{aligned} \tag{5.56}$$

Proof. The starting point is the simplest non-trivial identity of Theorem 5.2.8, namely $m = 2$ (Pfaffian of size 4), which gives

$$\begin{aligned}
&\bigg\langle \frac{\det(x_1-\Lambda)^2\det(x_2-\Lambda)}{\det(x_1'-\Lambda)^2\det(x_2'-\Lambda)^2}\bigg\rangle_N^V \\
&= \frac{N}{N+1}(x_1-x_1')(x_2-x_2')(x_1-x_2')(x_2-x_1')\frac{Z_{N+1}^V Z_{N-1}^V}{(Z_N^V)^2}\mathcal{K}_{N-1}^{\frac{N}{N-1}V}\big(\begin{smallmatrix} 2 & 2 \\ x_1 & x_2\end{smallmatrix}\big)\mathcal{K}_{N+1}^{\frac{N}{N+1}V}\big(\begin{smallmatrix} -2 & -2 \\ x_1' & x_2'\end{smallmatrix}\big) \\
&\quad - \frac{(x_1-x_2')(x_2-x_1')}{(x_1-x_2)(x_1'-x_2')}\mathcal{K}_N^V\big(\begin{smallmatrix} 2 & -2 \\ x_1 & x_1'\end{smallmatrix}\big)\mathcal{K}_N^V\big(\begin{smallmatrix} 2 & -2 \\ x_2 & x_2'\end{smallmatrix}\big) + \frac{(x_1-x_1')(x_2-x_2')}{(x_1-x_2)(x_1'-x_2')}\mathcal{K}_N^V\big(\begin{smallmatrix} 2 & -2 \\ x_1 & x_2'\end{smallmatrix}\big)\mathcal{K}_N^V\big(\begin{smallmatrix} 2 & -2 \\ x_2 & x_1'\end{smallmatrix}\big).
\end{aligned}$$

We omit the details of the asymptotic analysis based on Lemmata 5.4.8 and 5.4.9: it is very similar to the $\beta = 1$ case. Instead of using them for $K = 2N$, $p = \pm 2$ and $c, \tilde{c} \in \{-1, 1\}$, now we rather use them with $K = N$ and $p = \pm 1$ and $c, \tilde{c} \in \{-2, 2\}$. ☐

Lemma 5.5.8. *Theorem 5.5.7 is equivalent to Theorem 5.5.4.*

Proof. We apply Theorem 5.5.4 to the hyperelliptic curve with matrix of periods $\boldsymbol{\tau}' = -\boldsymbol{\tau}^{-1}$. Then, (5.50) is an identity involving theta functions with matrix $\frac{\boldsymbol{\tau}'}{2} = -\frac{\boldsymbol{\tau}^{-1}}{2}$. On the other hand, the modular transformation of the theta function is (see [Mum83, Equation 5.1]), for any $\boldsymbol{z}, \boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{R}^g$

$$\vartheta_{\boldsymbol{\nu},-\boldsymbol{\mu}}\big(\boldsymbol{z}\big| -\tfrac{\boldsymbol{\tau}^{-1}}{2}\big) = D_{\boldsymbol{\tau}} \cdot \mathrm{e}^{2\mathrm{i}\pi \boldsymbol{z}\cdot\boldsymbol{\tau}^{-1}(\boldsymbol{z})}\vartheta_{\boldsymbol{\mu},\boldsymbol{\nu}}\big(2\boldsymbol{z}\big|2\boldsymbol{\tau}\big).$$

for some constant $D_{\boldsymbol{\tau}} \in \mathbb{C}^*$. Applying this to each term in Theorem 5.5.7, all terms get the same prefactor and we are left with Theorem 5.5.7. The operation is reversible. ☐

5.5.4 Formula for the multi-cut equilibrium energy (Proof of Proposition 5.4.3)

In the proof of Theorem 5.5.1, if we did not use Proposition 5.4.3 to simplify the exponential in (5.45), the rest of the arguments would prove the identity (5.42) with a prefactor

$$e^{2\mathcal{E}[\mu_{\mathrm{eq}}]+2\mathcal{L}[V]+\mathcal{Q}[V,V]+4\mathrm{i}\pi\boldsymbol{\epsilon}^*\cdot(\boldsymbol{\tau}(\boldsymbol{\epsilon}^*)+\boldsymbol{u}(\infty_-))} \tag{5.57}$$

in the right-hand side, valid for any hyperelliptic curve with real Weierstraß points and the equilibrium measure μ_{eq} of the associated (unconstrained) $\beta = 2$ ensemble. Taking all points z, z', w, w' to ∞_+

in this modified identity implies that this extra factor (5.57) must be equal to 1. The argument of the exponential is manifestly real, except perhaps or the last term. As the curve is hyperelliptic, a basis of the space of holomorphic forms is given by $\mathrm{d}\pi_k = \frac{x^k \mathrm{d}x}{s}$ for $k \in [g]$. Recall that s takes imaginary values on the segments $[a_h, b_h]$ for each $h \in [0, g]$, and real values between the segments. This implies that the matrix $Q_{k,h} = \oint_{\mathcal{A}_h} \mathrm{d}\pi_k$ has purely imaginary entries. Since $(\mathrm{d}u_h)_{h=1}^{g}$ is the basis dual to $\mathcal{A}$-cycle integration, we have

$$\mathrm{d}u_h = \sum_{k=1}^{g} Q_{h,k}^{-1} \mathrm{d}\pi_k, \qquad \text{with } Q^{-1} \text{ purely imaginary.}$$

Integrating this on the $\mathcal{B}$-cycles which only run between segments (Section 5.3.3) yields a purely imaginary matrix of periods $\boldsymbol{\tau}$. A path from ∞_+ to ∞_- that does not cross any of the $\mathcal{A}$- and $\mathcal{B}$-cycles described in Section 5.3.3 is for instance the path travelling along the real axis in $\hat{C}_+$ from $-\infty$ to a_0, then along the real axis in $\hat{C}_-$ from a_0 to $-\infty_-$. In this range s is real-valued, so $\boldsymbol{u}(\infty_-)$ is also purely imaginary. All in all, (5.57) only involves the real exponential, and we conclude that

$$2\mathcal{E}[\mu_{\mathrm{eq}}] + 2\mathcal{L}[V] + \mathcal{Q}[V, V] + 4\mathrm{i}\pi\boldsymbol{\epsilon}^* \cdot (\boldsymbol{\tau}(\boldsymbol{\epsilon}^*) + \boldsymbol{u}(\infty_-)) = 0.$$

This argument was for $\beta = 2$, but we retrieve Proposition 5.4.3 in full generality since it is simply the $\beta = 2$ identity multiplied by $\frac{\beta}{2}$ and taking into account the prefactor $\frac{2}{\beta}$ in the definition of $\mathcal{Q}$, while μ_{eq} and $\mathcal{L}$ are independent of β. So, it was justified (without loop in the logic) to proceed with Proposition 5.4.3 in the proofs of Section 5.5. In fact, the same argument would establish Proposition 5.4.3 as a byproduct of the proof of the $\beta = 1$ Theorem 5.5.4 or of the $\beta = 4$ Theorem 5.5.7 instead of Theorem 5.5.1.

5.6 Variation of the entropy with respect to filling fractions (Proof of Proposition 5.4.4)

Consider the equilibrium measure $\mu_{\mathrm{eq},\boldsymbol{\epsilon}}$ of a β-ensemble with fixed filling fractions $\boldsymbol{\epsilon}$ such that $M(x) = t_{2g+2}\prod_{h=1}^{g}(x - z_h)$ with $z_h \in (b_{h-1}, a_h)$ in the notations of Section 5.2.3. The density of $\mu_{\mathrm{eq},\boldsymbol{\epsilon}}$ is

$$\rho(x) = \frac{t_{2g+2}}{2\pi} \prod_{h=1}^{g} |x - z_h| \prod_{h=0}^{g} \sqrt{|x - a_h||x - b_h|} \cdot \mathbb{1}_S(x).$$

We need to compute for each $h \in [g]$

$$v_{\mathrm{eq},h} = \Big(\frac{\beta}{2} - 1\Big) \int_S \partial_{\epsilon_h}\big(\rho(x)\ln\rho(x)\big) = \Big(\frac{\beta}{2} - 1\Big) \int_S \big(\partial_{\epsilon_h}\rho(x)\big)\ln\rho(x)\mathrm{d}x. \tag{5.58}$$

For the last equality we used that $\int_S \rho(x)\mathrm{d}x = 1$ has vanishing ϵ_h-derivative. The density ρ can be expressed as a jump of W_1 to rewrite

$$\begin{aligned} v_{\mathrm{eq},h} &= \Big(\frac{\beta}{2} - 1\Big) \int_S \partial_{\epsilon_h} \frac{W_1(x - \mathrm{i}0) - W_1(x + \mathrm{i}0)}{2i\pi} \ln\rho(x)\mathrm{d}x \\ &= \Big(\frac{\beta}{2} - 1\Big)\bigg(\sum_{k=1}^{g} \Upsilon_h(z_k) + \frac{1}{2}\sum_{h=0}^{g}\big(\Upsilon_h(a_h) + \Upsilon_h(b_h)\big)\bigg) \end{aligned} \tag{5.59}$$

in terms of the integrals

$$\forall \xi \in \mathbb{R} \qquad \Upsilon_h(\xi) := \int_S \partial_{\epsilon_h}\Big(\frac{W_1(x - \mathrm{i}0) - W_1(x + \mathrm{i}0)}{2\mathrm{i}\pi}\Big) \ln|x - \xi|\mathrm{d}x. \tag{5.60}$$

It is well-known (see *e.g.* [BG24, Appendix A]) that

$$\forall z \in \hat{C}_+ \qquad \partial_{\epsilon_h} W_1(X(z))\mathrm{d}X(z) = 2\mathrm{i}\pi \mathrm{d}u_h(z).$$

For $x \in \mathbb{C} \setminus S$ or in $S \pm \mathrm{i}0$, we define $\mathfrak{z}(x)$ to be the unique point in $\overline{\hat{C}_+}$ such that $X(\mathfrak{z}(x)) = x$. Then:

$$\Upsilon_h(\xi) = \int_S \big(\mathrm{d}u_h(\mathfrak{z}(x-\mathrm{i}0)) - \mathrm{d}u_h(\mathfrak{z}(x+\mathrm{i}0))\big)\ln|x-\xi| = 2\int_S \mathrm{d}u_h(\mathfrak{z}(x-\mathrm{i}0))\ln|x-\xi|.$$

This is a differentiable function of ξ. For $\xi \notin S$, we can compute

$$\partial_\xi \Upsilon_h(\xi) = \int_S \big(\mathrm{d}u_h(\mathfrak{z}(x-\mathrm{i}0) - \mathrm{d}u_h(\mathfrak{z}(x+\mathrm{i}0))\big)\frac{1}{\xi - x} = \oint_S \frac{\mathrm{d}u_h(z)}{\xi - X(z)} = 2\mathrm{i}\pi \frac{\mathrm{d}u_h}{\mathrm{d}X}(\mathfrak{z}(\xi)).$$

For $\xi \in \mathring{S}$, we rather have

$$\partial_\xi \Upsilon_h(\xi) = 2\int_S \frac{\mathrm{d}u_h(\mathfrak{z}(x-\mathrm{i}0))}{\xi - x} = -\frac{\mathrm{d}u_h}{\mathrm{d}X}(\mathfrak{z}(\xi+\mathrm{i}0)) - \frac{\mathrm{d}u_h}{\mathrm{d}X}(\mathfrak{z}(\xi-\mathrm{i}0)) = 0.$$

We will integrate this starting along the real line starting from $\xi = -\infty + \mathrm{i}0$ and using the continuity of Υ_h on the real axis shifted by $+\mathrm{i}0$. From the definition (5.60) we can see that $\lim_{\xi\to\infty}\Upsilon_h(\xi) = 0$. Therefore

$$\frac{\Upsilon_h(\xi)}{2\mathrm{i}\pi} = \begin{cases} u_h(\mathfrak{z}(\xi)) + \sum_{l=0}^{k-1}\big(u_h(a_l) - u_h(b_l)\big) & \text{if} \quad \xi \in (b_{k-1}, a_k), \\ u_h(a_k) + \sum_{l=0}^{k-1}\big(u_h(a_k) - u_h(b_k)\big) & \text{if} \quad \xi \in [a_k, b_k], \end{cases} \tag{5.61}$$

with the conventions $b_{-1} = -\infty$ and $a_{g+1} = +\infty$. Note that we could start integrating along the real line coming from $+\infty$, but we would get an equivalent expression because

$$\sum_{k=0}^{g} \boldsymbol{u}(a_k) = \sum_{k=0}^{g} \boldsymbol{u}(b_k). \tag{5.62}$$

The primitive $\boldsymbol{u}$ of $\mathrm{d}\boldsymbol{u}$ in $(\mathbb{C} \setminus S)$ is multivalued, because this domain is not simply-connected. Yet, for the previous computation, it suffices to define it by integration based at ∞_+ in the simply-connected domain $\mathbb{H} \setminus S$, and it is extended to S and hence $\overline{\mathbb{H}}$ by continuity. Inserting the formula (5.61) in (5.59) we arrive to

$$\boldsymbol{v}_{\mathrm{eq},h} = 2\mathrm{i}\pi\Big(\frac{\beta}{2}-1\Big)\Bigg[\sum_{k=1}^{g}\big(\boldsymbol{u}(z_k)+\boldsymbol{u}(a_0)-\boldsymbol{u}(b_0)+\cdots+\boldsymbol{u}(a_{k-1})-\boldsymbol{u}(b_{k-1})\big)+\sum_{k=0}^{g}\frac{\boldsymbol{u}(a_k)+\boldsymbol{u}(b_k)}{2}\Bigg]. \tag{5.63}$$

We now compute $\boldsymbol{u}(a_k)$ and $\boldsymbol{u}(b_k)$ as defined above. Denote $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_g)$ the canonical basis of $\mathbb{C}^g$. Due to the description of the representatives of the $\mathcal{A}$- and $\mathcal{B}$-cycles in Section 5.3.3 and the fact that the hyperelliptic involution changes the sign of $\mathrm{d}\boldsymbol{u}$, we have

$$\boldsymbol{u}(b_0) - \boldsymbol{u}(a_0) = -\frac{1}{2}\oint_{\mathcal{A}_0}\mathrm{d}\boldsymbol{u} = \frac{1}{2}\sum_{l=1}^{g}\boldsymbol{e}_l, \tag{5.64}$$

and for any $k \in [g]$

$$\begin{aligned} \boldsymbol{u}(b_k) - \boldsymbol{u}(a_k) &= -\frac{1}{2}\oint_{\mathcal{A}_k}\mathrm{d}\boldsymbol{u} = -\frac{1}{2}\boldsymbol{e}_k, \\ \boldsymbol{u}(a_k) - \boldsymbol{u}(b_{k-1}) &= \frac{1}{2}\oint_{\mathcal{B}_k - \mathcal{B}_{k-1}}\mathrm{d}\boldsymbol{u} = \frac{1}{2}\big(\boldsymbol{\tau}(\boldsymbol{e}_k) - \boldsymbol{\tau}(e_{k-1})\big), \end{aligned} \tag{5.65}$$

with the conventions $\mathcal{B}_0 = 0$ and $\boldsymbol{e}_0 = 0$. Since a_0 is the only Weierstraß point that does not belong to the $\mathcal{A}$- and $\mathcal{B}$-cycles specified in Section 5.3.3, $\boldsymbol{u}(\infty_-)$ can be obtained by integrating $\mathrm{d}\boldsymbol{u}$ in the first sheet $-\infty$ on the real line to a_0, and then to a_0 from $-\infty$ on the real line in the second sheet. Therefore

$$\boldsymbol{u}(a_0) = \frac{1}{2}\boldsymbol{u}(\infty_-).$$

From (5.64)-(5.65) we deduce

$$\boldsymbol{u}(b_0) = \frac{1}{2}\Big(\boldsymbol{u}(\infty_-) + \sum_{l=1}^{g} \boldsymbol{e}_l\Big),$$

and for $k \in [g]$

$$\boldsymbol{u}(a_k) = \frac{1}{2}\Big(\boldsymbol{u}(\infty_-) + \sum_{l=k}^{g} \boldsymbol{e}_l + \sum_{l=1}^{k} \boldsymbol{\tau}(\boldsymbol{e}_l)\Big),$$
$$\boldsymbol{u}(b_k) = \frac{1}{2}\Big(\boldsymbol{u}(\infty_-) + \sum_{l=k+1}^{g} \boldsymbol{e}_l + \sum_{l=1}^{k} \boldsymbol{\tau}(\boldsymbol{e}_l)\Big).$$

Therefore

$$\begin{aligned}\sum_{k=0}^{g} \boldsymbol{u}(a_k) = \sum_{k=0}^{g} \boldsymbol{u}(b_k) &= \frac{1}{2}\bigg[(g+1)\boldsymbol{u}(\infty_-) + \sum_{k=1}^{g}\bigg(\sum_{l=k}^{g} \boldsymbol{e}_l + \sum_{l=1}^{k} \boldsymbol{\tau}(\boldsymbol{e}_l)\bigg)\bigg] \\ &= \frac{1}{2}\Big((g+1)\boldsymbol{u}(\infty_-) + \sum_{l=1}^{g}\big(l\boldsymbol{e}_l + (g+1-l)\boldsymbol{\tau}(\boldsymbol{e}_l)\big)\Big).\end{aligned}$$

We can return to the computation of $\boldsymbol{v}_{\mathrm{eq}}$. By definition in (5.31) it is real, so we can replace $\boldsymbol{u}$ by $\mathrm{Im}\,\boldsymbol{u}$ in (5.63). Since $\boldsymbol{u}(b_l) - \boldsymbol{u}(a_l)$ is real for any $l \in [0, g]$, we get

$$\boldsymbol{v}_{\mathrm{eq}} = 2\pi\Big(1 - \frac{\beta}{2}\Big)\bigg[\sum_{k=1}^{g}\Big(\mathrm{Im}\,\boldsymbol{u}(z_k) + \frac{g+1-k}{2}\,\mathrm{Im}\,\boldsymbol{\tau}(e_k)\Big) + \frac{g+1}{2}\mathrm{Im}\,\boldsymbol{u}(\infty_-)\bigg].$$

Since we already know that $\boldsymbol{\tau}$ and $\boldsymbol{u}(\infty_-)$ are purely imaginary, we can drop imaginary part and divide by i instead, and this is the final formula.

Chapter 6

Semi-localization in the Generalized Random Graph model

This Chapter is based on a joint work with Antti Knowles [BK24].

6.1 Introduction

This Chapter is devoted to showing a semi-localization result for the adjacency matrix of the Generalized Random Graph (GRG) model. The properties of this random graph model are studied in great details in [vdHof16, Chapter 6]. Let us recall the definition of the model. Let $N \geq 1$ be an integer. We consider a graph $G = ([N], E)$, where $[N] = \{1, \dots, N\}$ is the set of vertices of G and $E \subset \{\{x, y\}, x, y \in [N], x \neq y\}$ is the set of edges. We take G to be a random graph, distributed according to the Generalized Random Graph (GRG) model, defined as follows. To each vertex $x \in [N]$, we attached a weight $w_x > 0$. Thus, $\boldsymbol{w}^N = (w_x)_{x\in[N]}$ is a sequence of weights that depends on N. We shall omit this dependence from the notation in the sequel. Two distinct vertices x and y are connected in G, i.e. $\{x, y\} \in E$, with probability

$$p_{xy} = \mathbb{P}\left(\{x, y\} \in E\right) = \frac{w_x w_y}{\sum_{z\in[N]} w_z + w_x w_y},$$

and the random variables $\left(\mathbb{1}_{\{\{x,y\}\in E\}}\right)_{x,y\in[N]}$ are independent.

We denote by A the adjacency matrix of G, with coefficients $A_{xy} = \mathbb{1}_{\{\{x,y\}\in E\}}$. We endow G with the usual graph distance. Let $x \in [N]$ and $r \geq 1$. We denote by $B_r(x)$ (respectively $S_r(x)$) the set of vertices at distance at most (resp. exactly) r of the vertex x. The degree of a vertex x is denoted by

$$D_x = \#S_1(x) = \sum_{y\in[N]} \mathbb{1}_{\{x\sim y\}}.$$

In the sequel, we will write the event $\{\{x, y\} \in E\}$ as $\{x \sim y\}$.

In this article, we make two hypotheses on the sequence of weights. To state them, we introduce the **empirical moments**

$$m_\alpha = m_{N,\alpha} = \frac{1}{N}\sum_{x=1}^{N} w_x^\alpha. \tag{6.1}$$

Notation 6.1.1. We fix for the rest of the article two parameters $\epsilon > 0$ and $0 < \delta < 1/3$ such that Hypotheses 6.1.2 and 6.1.3 below are satisfied.

Hypothesis 6.1.2. *For all* $x \in [N]$,

$$w_x \leq N^{1/2-\epsilon}.$$

Hypothesis 6.1.3. *The first and second empirical moments satisfy*

$$m_1 \geq N^{-\epsilon} \quad \textit{and} \quad \frac{m_2}{m_1} = \mathcal{O}\Big((\ln N)^{\delta}\Big).$$

Remark 6.1.4. We have the simple bound

$$p_{xy} = \frac{w_x w_y}{m_1 N + w_x w_y} \leq \frac{w_x w_y}{m_1 N}. \tag{6.2}$$

Furthermore, Hypotheses 6.1.2 and 6.1.3 imply that for all $x, y \in [N]$,

$$p_{xy} = \frac{w_x w_y}{N m_{N,1} + w_x w_y} = \frac{w_x w_y}{N m_1 + \mathcal{O}(N^{1-2\epsilon})} = \frac{w_x w_y}{m_1 N}\left(1 + \mathcal{O}\left(\frac{N^{-2\epsilon}}{m_1}\right)\right) = \frac{w_x w_y}{m_1 N}\left(1 + \mathcal{O}(N^{-\epsilon})\right).$$

We are thus considering the regime where the GRG model and the Chung-Lu model coincide approximately. Indeed, recall that in the Chung-Lu model,

$$p_{xy}^{\mathrm{Chung-Lu}} = \frac{w_x w_y}{m_1 N} \wedge 1.$$

Given a sequence of probability distributions $(\mu_N)_{N \geq 1}$, there are two natural ways to define the sequence of weights:

- the weight sequence $\boldsymbol{w}^N = (w_x)_{x \in [N]}$ can be sampled according to $\mu_N^{\otimes N}$,
- the weights $\boldsymbol{w}^N = (w_x)_{x \in [N]}$ can be chosen to be the $N+1$-quantiles of μ_N.

We now present some examples in which our Theorem 6.1.9, stated below, is applicable.

Example 6.1.5 (Exponential distribution). Assume that μ_N is the exponential distribution with parameter $\alpha > 0$. For all constant parameter $\alpha > 0$, Hypotheses 6.1.2 and 6.1.3 are satisfied with high probability in the i.i.d weight case, or always in the quantile case. In the latter case, the weights are chosen to be

$$w_k = \frac{1}{\alpha} \ln \frac{N}{k+1},$$

for $1 \leq k \leq N$.

If $\alpha < 1/(2\nu)$, Lemma 6.1.10 implies that the greatest degree in the graph is of order $\ln N$, and our result, Theorem 6.1.9 below, is not vacuous.

Example 6.1.6 (A model with heavy tails for the weights). Assume that μ_N is the measure having density with respect to the Lebesgue measure

$$\frac{\mathrm{d}\mu_N}{\mathrm{dLeb}}(t) = \frac{\alpha^2}{\alpha - 1} d^{-1} \left(\frac{t}{t_0}\right)^{-1-\alpha} \mathbb{1}_{(t_0, +\infty)}(t),$$

with $t_0 = \frac{\alpha-1}{\alpha} d$, and $\alpha > 2$. The constants are chosen so that this is a probability density with expectation d. The choice $\alpha > 2$ ensures that a second moment exists and is finite, and hence that Hypothesis 6.1.3 can be satisfied.

When taking weights to be i.i.d samples, Hypotheses 6.1.2 and 6.1.3 are not satisfied in general. However, the event that these two hypotheses are satisfied is an event of high probability. When taking the weights to be N-quantiles of μ_N, the weight of vertex $x \in [N]$ is

$$w_x = t_0 \left(\frac{N}{x}\right)^{1/\alpha}.$$

In that case, both Hypotheses 6.1.2 and 6.1.3 are satisfied when $\alpha > 2$ and $d \geq 1$.

As shown in [vdHof16, Theorem 6.12], the choice of such a heavy tailed distribution for the weights implies that the vertex degrees in the graph will have a heavy tailed distribution as well.

We will derive results which hold with very high probability, in the following sense.

Definition 6.1.7. *Let $\nu > 0$. A sequence of events (E_N) is said to be satisfied with ν-high probability if there exists $C > 0$ such that*

$$\mathbb{P}(E_N) \geq 1 - CN^{-\nu}.$$

6.1.1 Semi-localization result

We now state a semi-localization result concerning the eigenvectors of A associated to its largest eigenvalues. Such an eigenvector is shown to be essentially supported on vertices of high degree. We define the set of *resonant vertices* by

$$\mathcal{W}_{\lambda,\eta} = \{x \in [N] \colon |\sqrt{D_x} - \lambda| \leq \eta\}.$$

The set of resonant vertices is in cases of interest of negligible size compared to the size of the graph.

Proposition 6.1.8. *Let $\nu > 0$. There exists $C_\nu > 0$ such that with ν-high probability, the following statement is true: for all $C_\nu \sqrt{\ln N / \ln \ln N} < \eta < \lambda/2$, we have*

$$\#\mathcal{W}_{\lambda,\eta} \leq 2\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta+1}\right] \vee \frac{2\nu \ln N}{\ln \ln N} \leq \frac{2m_2}{(\lambda-\eta)^4} N \vee \frac{2\nu \ln N}{\ln \ln N}.$$

Furthermore, when the weights are taken to be the quantile of the exponential or the heavy tail distribution as in Examples 6.1.5 and 6.1.6, we have respectively

$$\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta+1}\right] = \mathcal{O}\left(\frac{N}{(\alpha+1)^{(\lambda-\eta)^2}} + (\ln N)^{2\delta}\right),$$

and

$$\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta+1}\right] = \mathcal{O}\left(\frac{N\eta}{\lambda^{2\alpha+3}} + (\ln N)^{2\delta}\right).$$

The main result is then as follows.

Theorem 6.1.9. *Let $\nu > 0$. There exists $C_\nu > 0$ such that for all N the following statement holds with ν-high probability. For all eigenvalue λ with associated normalized eigenvector q, and for all $\eta \leq |\lambda|/2$, we have*

$$\sum_{x \in \mathcal{W}_{\lambda,\eta}} \langle q, u_{\operatorname{sign}\lambda}(x)\rangle^2 \geq 1 - \frac{C_\nu}{\eta^2}\frac{\ln N}{\ln \ln N},$$

where $u_\pm(x)$ is a vector supported in $B_2(x)$, defined in Proposition 6.3.4.

6.1.2 The degrees and the weights

We give a few more results and definitions regarding the link between weights and degrees in G. In the sequel, we will mainly consider vertices of high degree. In particular, vertices with weights greater than $c \ln N$, for a fixed $c > 0$, are easier to describe. The degree D_x of a vertex x concentrates around its expectation d_x

$$d_x = \mathbb{E}\left[D_x\right] = \sum_{y \in [N]\setminus\{x\}} \frac{w_x w_y}{\sum_z w_z + w_x w_y} \leq w_x.$$

Remark 6.1.4 implies that

$$d_x = \sum_{y \in [N]\setminus\{x\}} \frac{w_x w_y}{m_1 N}(1+o(1)) = w_x(1 - \frac{w_x}{m_1 N})(1+o(1)) = w_x(1+o(1)). \tag{6.3}$$

As soon as the weight of a vertex x is at least of order $\ln N$, the degree D_x is as well at least of order $\ln N$, with very high probability.

Lemma 6.1.10. *Let $\nu > 0$ and $x \in [N]$. With ν-high probability,*

$$d_x - \sqrt{2\nu d_x \ln N} \leq D_x \leq d_x + 2\sqrt{\nu \ln N (d_x \vee \frac{4\nu}{9} \ln N)}.$$

Proof. This is an application of Bennett's inequality (see [BLM13, Theorem 2.9]). We have for $M > 0$,

$$\mathbb{P}(D_x \geq M + d_x) = \mathbb{P}(D_x - d_x \geq M) \leq \exp\left(-\frac{M^2}{2(w_x + M/3)}\right).$$

If we choose $M = 2\sqrt{\nu \ln N (w_x \vee (2/3)\nu \ln N)}$, we get the upper bound.

For the lower bound, we use the slightly sharper bound (see [vdHof16, Theorem 2.21]),

$$\mathbb{P}(D_x \leq d_x - \sqrt{2\nu d_x \ln N}) \leq \exp\left(-\frac{(\sqrt{2\nu d_x \ln N})^2}{2d_x}\right) \leq N^{-\nu}.$$

□

In the large N limit, most of the random variables we consider have Poisson tails. The $\ln N$ scale thus appears naturally when deriving results that hold with very high probability.

In the sequel, it will be convenient to order the vertices in terms of their degree.

Definition 6.1.11. *We define the order relation $\prec$ on the set of vertices $[N]$ as follows. For any two vertices $x, y \in [N]$,*

$$x \prec y \text{ if and only if } ((D_x < D_y) \text{ or } (D_x = D_y \text{ and } x > y)).$$

We define the (random) permutation $\pi \in \mathfrak{S}_N$ to be the unique permutation such that

$$\pi(N) \prec \pi(N-1) \prec \cdots \prec \pi(2) \prec \pi(1).$$

Note that in particular $D_{\pi(N)} \leq D_{\pi(N-1)} \leq \cdots \leq D_{\pi(2)} \leq D_{\pi(1)}$. This order allows us to define two notions of neighborhood and degree:

$$\begin{aligned} S_1^+(x) &= \{y \in [N] \colon x \sim y, x \prec y\} \text{ and } D_x^+ = \#S_1^+(x), \\ S_1^-(x) &= \{y \in [N] \colon x \sim y, x \succ y\} \text{ and } D_x^- = \#S_1^-(x). \end{aligned} \tag{6.4}$$

6.2 Pruning the graph

Similarly as in [ADK21b; ADK21a], it is more convenient to work on a pruned version of the graph. The GRG model is inhomogeneous, compared to the Erdös-Rényi model: there are greater differences of degrees in the graph. Because of this greater heterogeneity, the pruning procedure has to be more subtle. As in the Erdös-Rényi case, we first prune the graph to remove cycles in small balls. Then, instead of removing all edges connecting two vertices of high degree, we remove edges appearing in a very specific pattern. This procedure is key to simplifying the computations in Section 6.3.

We fix $r \geq 6$. For convenience, we will sometimes omit it from the notation.

Remark 6.2.1. The pruning of [ADK21a] (besides removing cycles) amounts to to remove all the edges between vertices of high degree. Because of the inhomogeneity in the GRG case, this would mean cutting a number of vertices proportional to w_x around a vertex x. This would prevent us from obtaining good bounds on the error we make when replacing the adjacency matrix of the original graph by the one of the pruned graph.

6.2.1 The set of vertices of high degree

We will be interested in the vertices with the greatest degrees in the graph G.

Definition 6.2.2. *We define the threshold*

$$\xi = \xi_\nu = \frac{4(\nu+1)(2-3\delta)}{(1-\delta)(1-2\delta)} \frac{\ln N}{\ln \ln N}.$$

We define two sets of vertices of high degree

$$\mathcal{V} = \left\{ x \in [N] \colon D_x > \frac{\ln N}{\ln \ln N} \right\}, \text{ and}$$
$$\mathcal{V}_\nu = \{ x \in [N] \colon D_x > \xi_\nu \}.$$

Lemma 6.2.3. *Let $\nu > 0$ and $C > 0$. Let $x \in [N]$. Then, with ν-high probability,*

$$(4\nu \ln N)\, D_x \geq d_x.$$

Furthermore, if $D_x \geq C \frac{\ln N}{\ln \ln N}$ then with ν-high probability

$$\left(\frac{4\nu}{C} \ln \ln N \right) D_x \geq d_x.$$

Proof. Assume first that $d_x > 4\nu \ln N$. By Lemma 6.1.10 we have with ν-high probability that

$$D_x \geq d_x - \sqrt{2\nu d_x \ln N}.$$

With $d_x > 4\nu$, this yields

$$D_x \geq d_x \left(1 - \sqrt{1/2} \right),$$

and the two claims follow.

If $d_x \leq 4\nu \ln N$, we get the first claim as $D_x \geq 1$. Assuming $D_x \geq C \frac{\ln N}{\ln \ln N}$ we get

$$\frac{d_x}{D_x} \leq 4\nu \ln N \frac{\ln \ln N}{C \ln N} = \frac{4\nu}{C} \ln \ln N,$$

hence the second claim. □

6.2.2 Removing the cycles

In this Section, we estimate the number of edges to prune around each vertex to remove all cycles in small balls centered around each vertex. To state this precisely we give a few definitions about paths.

Definition 6.2.4. *A **path** γ in the graph G is a sequence of vertices $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_l)$, with $\{\gamma_{i-1}, \gamma_i\} \in E$ for $i \in [n]$. The length of the path is $l(\gamma) = l$.*

*A path is said to be **simple** if $\gamma_i \neq \gamma_j$ for $i \neq j, \{i, j\} \neq \{0, l\}$. Two paths γ and γ' are **edge-intersecting** if there exists i and j such that $\gamma_i = \gamma'_j$ and either $\gamma_{i-1} = \gamma'_{j-1}$ or $\gamma_{i-1} = \gamma'_{j+1}$.*

The set of paths in G is denoted by $\mathcal{P}(G)$, the set of simple paths in G is denoted by $\mathcal{P}^*(G)$, and the set of simple paths with length less than l is denoted by $\mathcal{P}^*_{\leq l}(G)$.

Consider a vertex $x \in [N]$. We define

$$S_1^{\mathrm{cyc}}(x) = \{ y \in S_1(x) \colon \exists \gamma \in \mathcal{P}^*_{\leq 2r+1}(G), \gamma_0 = \gamma_{l(\gamma)} = x, \gamma_1 = y \},$$

i.e. the set of vertices connected to x that are part of a cycle which is a simple loop. In Section 6.2.4 we will prune the edges in the set $\{\{x, y\} \colon y \in S_1^{\mathrm{cyc}}(x)\}$ so as to remove all cycles in all small balls. In this Section, we give bound for the cardinality of the set $S_1^{\mathrm{cyc}}(x)$.

Proposition 6.2.5. *Fix $x \in [N]$ and $\nu > 0$. There exists a constant C_ν such that with ν-high probability,*

$$\# S_1^{\mathrm{cyc}}(x) \leq C_\nu.$$

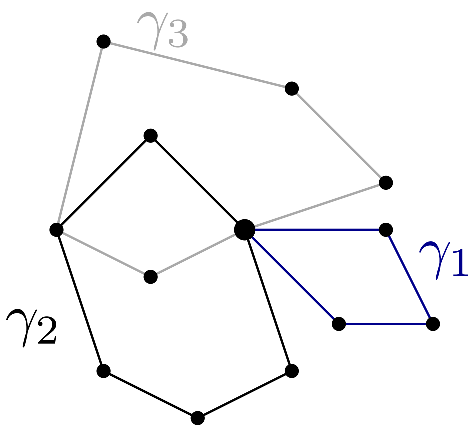


Figure 6.1: A representation of the event $\mathrm{NE}_{k,r}$.
The simple paths can share a vertex, but cannot e edge-intersecting.

We shall consider the following event concerning non edge-intersecting paths

$$\mathrm{NE}_{k,r}(x) = \left\{ \exists \gamma^{(1)}, \dots, \gamma^{(k)} \in \mathcal{P}^*(G|_{B_r(x)}) : \begin{array}{l} \forall i \neq j, \gamma^{(i)} \text{ and } \gamma^{(j)} \text{ are not edge-intersecting,} \\ \forall i, \gamma^{(i)}_0 = \gamma^{(i)}_{l(\gamma^{(i)})} = x, \\ \sum_{i=1}^{k} l(\gamma^{(i)}) \leq 2k(2r+1) \end{array} \right\}.$$

It is depicted in Figure 6.1.

We argue that the cardinal $\#S_1^{\mathrm{cyc}}(x)$ can be bounded by the number of non edge-intersecting paths in the graph $G|_{B_r(x)\setminus\{x\}}$ between pairs of points of $S_1(x)$.

Lemma 6.2.6. *Let $k \geq 1$ and $x \in [N]$. We have the inclusion of events*

$$\{\#S_1^{\mathrm{cyc}}(x) \geq 2k\} \subset \mathrm{NE}_{k,r}(x).$$

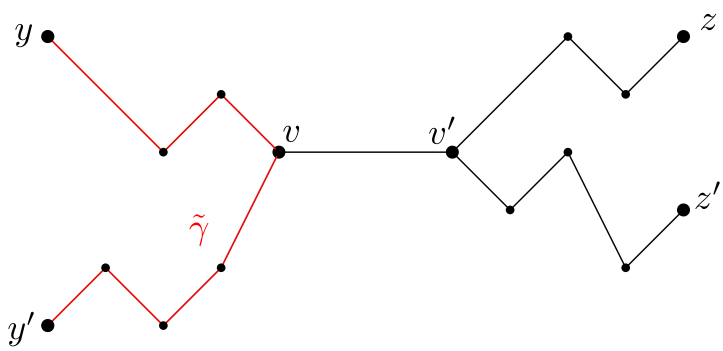


Figure 6.2: Construction of the path $\tilde{\gamma}$.
The paths γ and γ' are edge-intersecting, respectively from y to z and y' to z'.

Proof. Let $y_1, \dots, y_k \in S_1^{\mathrm{cyc}}(x)$ be distinct vertices. By definition, there exists $z_1, \dots, z_k \in S_1^{\mathrm{cyc}}(x)$, with $z_i \neq y_i$, such that there exits paths $\gamma_i, i = 1, \dots, k$, each of them connecting y_i to z_i (without going through x). We now argue that we can choose the paths γ_i such that they are either equal, or are not edge-intersecting.

Indeed, consider γ connecting $y \in S_1^{\mathrm{cyc}}(x)$ and $z \in S_1^{\mathrm{cyc}}(x)$, and γ' connecting $y' \in S_1^{\mathrm{cyc}}(x)$ and $z' \in S_1^{\mathrm{cyc}}(x)$. If γ and γ' have an edge $\{v, v'\}$ in common, with

$$\begin{cases} \gamma &= (\gamma_0 = y, \gamma_1, \gamma_2, \dots, \gamma_i = v, \gamma_{i+1} = v', \gamma_{i+2}, \dots, \gamma_{l-1}, z) \\ \gamma' &= (\gamma'_0 = y', \gamma'_1, \gamma'_2, \dots, \gamma'_j = v, \gamma'_{j+1} = v', \gamma'_{j+2}, \dots, \gamma'_{l'-1}, z'), \end{cases}$$

i.e. the edge $\{v, v'\}$ is traversed in the same direction for both paths, then we see that if we consider the graph with the edge $\{v, v'\}$ removed, y is still connected to a distinct vertex of $S_1^{\text{cyc}}(x)$, that is y'. In this case, we obtain a paths $\tilde{\gamma}$, that connects y and y'.

If γ' is oriented otherwise, and v is encountered before v' in γ', we proceed differently. We do not remove the edge $\{v, v'\}$ from the paths. Instead, we construct one path $\tilde{\gamma}$ from these two paths

$$\tilde{\gamma} = (y, \gamma_1, \dots, \gamma_i = v, \gamma_j = v', \gamma'_{j-1}, \dots, \gamma'_1, y').$$

This construction is depicted in Figure 6.2.

To construct a family of non edge-intersecting paths from the paths going from the y_i's to the z_i's, we consider first the path $\gamma^{(1)}$ from y_1 to z_1. We consider the first edge e in γ_1 that intersect another path $\gamma^{(i)}$, $i \neq 1$. We apply the procedure described above, and obtain one path $\tilde{\gamma}^{(1)}$ whose first vertex is y_1 and last vertex is y_i. We obtain a new family of paths, $(\tilde{\gamma}^{(1)}, \gamma^{(2)}, \dots, \gamma^{(i-1)}, \gamma^{(i+1)}, \dots, \gamma^{(k)})$.

We keep applying this procedure on the first path of the family until it is no longer edge-intersecting with any other path of the family. We then consider the second path, and proceed as previously until all the paths are either non edge-intersecting, or identical up to orientation. Notice that this procedure terminates as there is a finite number of edges that are part of two or more paths.

At the end of the procedure, we have a family of paths, and each vertex y_i, $i = 1, \dots, k$ is at one end of at least one path paths. We thus have $k/2$ simple paths that are pairwise non edge-intersecting.

Notice that we have not added any edges in the procedure, thus the total length of the non edge-intersecting paths thus created is less than $k(2r-1)$. This shows that the required inclusion holds. □

Lemma 6.2.6 then implies the Proposition 6.2.5.

Proof of Proposition 6.2.5. Let $k \geq 1$. Lemma 6.2.6 implies that

$$\mathbb{P}\left(\#S_1^{\text{cyc}}(x) \geq 2k\right) \leq \mathbb{P}\left(\text{NE}_{k,r}(x)\right).$$

The union bound then implies

$$\mathbb{P}\left(\text{NE}_{k,r}(x)\right) \leq \sum_{\substack{l_1,\dots,l_k \geq 1 \\ \sum_i l_i \leq 2k(2r+1)}} \prod_{i=1}^{k} \left(\sum_{y_1,\dots,y_{l_i-1}} p_{xy_1} p_{y_1y_2} \cdots p_{y_{l_i-2} l_i-1} p_{l_i-1 x} \right).$$

Notice that we have independence of the edges because we ensured that the paths are not edge-intersecting.

We have

$$\sum_{y_1,\dots,y_{l_i-1}} p_{xy_1} p_{y_1y_2} \cdots p_{y_{l_i-2} l_i-1} p_{l_i-1 x} \leq \sum_{y_1,\dots,y_{l_i-1}} \frac{w_x^2 w_{y_1}^2 \cdots w_{l_i-1}^2}{m_1^{l_i} N^{l_i}} = \frac{w_x^2}{m_1 N} \left(\frac{m_2}{m_1}\right)^{l_i-1}.$$

Hypotheses 6.1.2 and 6.1.3 then imply

$$\sum_{y_1,\dots,y_{l_i-1}} p_{xy_1} p_{y_1y_2} \cdots p_{y_{l_i-2} l_i-1} p_{l_i-1 x} = \mathcal{O}\left(N^{-\epsilon/2}\right)$$

Finally, we have

$$\mathbb{P}\left(\#S_1^{\text{cyc}}(x) \geq 2k\right) \leq \mathcal{O}\left(N^{-k\epsilon/2}\right).$$

Choosing k big enough gives the result. □

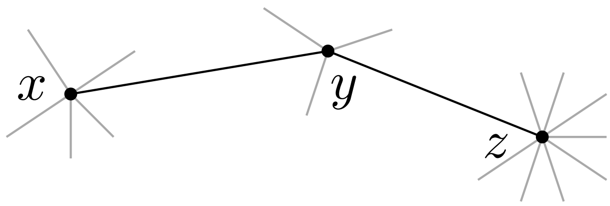


Figure 6.3: A down-up path.

6.2.3 Removing the down-up paths

In this Section, we will use the order defined in Definition 6.1.11.

Definition 6.2.7. *A down-up path between two distinct vertices $x \in [N]$ and $z \in [N]$ is a path (x, y, z) with $y \prec x \prec z$. The set of down-up vertices near x is*

$$S_1^{\mathrm{du}}(x) = \{y \colon \exists z, x \sim y \sim z, y \prec x \prec z\}.$$

We will remove all down-up paths in G. To do so we proceed as follows. Consider a vertex $x \in [N]$, and $y \in S_1(x)$ with $y \prec x$. If there exists $z \in S_1(y)$ with $z \succ x$, then we remove the edge $\{x, y\}$. To bound the number of edges removed around x, we use Lemma 6.2.8. Around y, this procedure removes exactly $(D_y^+ - 1) \vee 0$ (recall (6.4)) edges. This quantity is bounded using Lemma 6.2.9, below.

Lemma 6.2.8. *Let $\nu > 0$ and $x \in [N]$. With ν-high probability,*

$$\#\left(S_1^{\mathrm{du}}(x) \setminus S_1^{\mathrm{cyc}}(x)\right) \leq \frac{2\nu}{1-2\delta} \frac{\ln N}{\ln \ln N}.$$

Lemma 6.2.9. *Let $x \in [N]$ and $\nu > 0$. With ν-high probability,*

$$D_x^+ \leq \frac{3\nu}{2(1-\delta)} \frac{\ln N}{\ln \ln N}.$$

Thus, Lemmata 6.2.8 and 6.2.9 show that we will remove roughly $\ln N / \ln \ln N$ edges around each vertex when removing the down-up paths.

Proof of Lemma 6.2.8. Recall that

$$\#\left(S_1^{\mathrm{du}}(x) \setminus S_1^{\mathrm{cyc}}(x)\right) = \#\{y \in S_1(x) \setminus S_1^{\mathrm{cyc}}(x) \colon \exists z \in S_1(y) \setminus \{x\}, D_y \leq D_x \leq D_z\}.$$

Let $k \geq 1$. We have by the union bound,

$$\begin{aligned}
\mathbb{P}\Big(\#\big(S_1^{\mathrm{du}}(x) \setminus S_1^{\mathrm{cyc}}(x)\big) \geq k \mid S_1(x)\Big) &= \mathbb{P}\left(\exists I \subset \left(S_1^{\mathrm{du}}(x) \setminus S_1^{\mathrm{cyc}}(x)\right) : |I| = k \mid S_1(x)\right) \\
&\leq \sum_{\substack{I \subset [N] \\ |I| = k}} \sum_{\{z_y; y \in I\}} \mathbb{P}\left(\forall y \in I, y \notin S_1^{\mathrm{cyc}}(x), x \sim y \sim z_y, D_x \leq D_{z_y} \mid S_1(x)\right) \\
&\leq \frac{1}{k!} \sum_{\substack{y_1, \dots, y_k, \\ \text{distinct}}} \sum_{z_1, \dots, z_k} \mathbb{P}\left(\forall i \in [k], y_i \notin S_1^{\mathrm{cyc}}(x), x \sim y_i \sim z_i, D_x \leq D_{z_i} \mid S_1(x)\right).
\end{aligned}$$

Notice that we removed the event $D_y \leq D_x$.

Notice that if $y_i \sim z_j$ for $i \neq j$, then $x \sim y_i \sim z_j \sim y_j \sim x$ is a cycle and $y_i \in S_1^{\mathrm{cyc}}(x)$. It is also the case if $y_i = z_j$ for some pair (i, j), if $z_i \sim z_j$, or if $x \sim z_i$. Thus, the vertices $y_1, \dots, y_k, z_1, \dots, z_k$ can be assumed to be distinct. Furthermore, if we define the random variable

$$\hat{D}_{z_i} = \#\left(S_1(z_i) \setminus \{y_1, z_1, \dots, y_k, z_k\}\right) = \sum_{z' \notin \{y_j, z_j; j \in [k]\}} \mathbb{1}_{\{z_i \sim z'\}},$$

we have the inclusion of events

$$\begin{Bmatrix} y_1, \dots, y_k \notin S_1^{\mathrm{cyc}}(x), \\ \forall i, x \sim y_i \sim z_i, D_x \le D_{z_i} \end{Bmatrix} \subset \begin{Bmatrix} y_1, \dots, y_k, z_1, \dots, z_k \text{ distinct} \\ \forall i, x \sim y_i \sim z_i, D_x \le \hat{D}_{z_i} + 1 \end{Bmatrix}.$$

Notice that the random variables $\hat{D}_{z_i}$ for $i \in [k]$ are independent conditionally to $S_1(x)$. It follows that

$$\mathbb{P}\left(\# \left(S_1^{\mathrm{du}}(x) \setminus S_1^{\mathrm{cyc}}(x)\right) \ge k \mid S_1(x)\right) \le \frac{1}{k!} \sum_{\substack{y_1, z_1, \dots, y_k, z_k \\ \text{distinct}}} \prod_{i=1}^{k} \mathbb{1}_{\{x \sim y_i\}} p_{y_i, z_i} \mathbb{P}\left(D_x \le \hat{D}_{z_i} + 1 \mid S_1(x)\right).$$

Markov's inequality and (6.2) give

$$\mathbb{P}\left(\# \left(S_1^{\mathrm{du}}(x) \setminus S_1^{\mathrm{cyc}}(x)\right) \ge k \mid S_1(x)\right) \le \frac{1}{k!} \sum_{\substack{y_1, z_1, \dots, y_k, z_k \neq x \\ \text{distinct}}} \prod_{i=1}^{k} \mathbb{1}_{\{x \sim y_i\}} \frac{w_{y_i} w_{z_i}}{m_1 N} \frac{d_{z_i} + 1}{D_x}.$$

Now, notice that if there exists k distinct vertices $y_1, \dots, y_k \in S_1(x)$, then $D_x \ge k$. Take $k = \lceil \frac{2\nu}{1-2\delta} \frac{\ln N}{\ln \ln N} \rceil$. Then, using Lemma 6.2.3 we have that with ν-high probability,

$$\mathbb{P}\left(\# \left(S_1^{\mathrm{du}}(x) \setminus S_1^{\mathrm{cyc}}(x)\right) \ge k \mid S_1(x)\right) \le \frac{(2 \ln \ln N)^k}{k!} \sum_{\substack{y_1, z_1, \dots, y_k, z_k \\ \text{distinct}}} \prod_{i=1}^{k} \mathbb{1}_{\{x \sim y_i\}} \frac{w_{y_i} w_{z_i} (w_{z_i} + 1)}{d_x m_1 N}.$$

Taking the expectation and using (6.3), that is $w_x = d_x(1 + o(1))$, as well as twice (6.1), we obtain

$$\mathbb{P}\left(\# \left(S_1^{\mathrm{du}}(x) \setminus S_1^{\mathrm{cyc}}(x)\right) \ge k\right) \le \frac{(2 \ln \ln N)^k}{k!} \left(\frac{m_2}{m_1} + 1\right)^{2k} (1 + o(1)) + \mathcal{O}(N^{-\nu}).$$

Using Stirling's asymptotics and Hypothesis 6.1.3 yields the result. ☐

Proof of Lemma 6.2.9. Consider the random variable

$$D_x^{\mathrm{nc}+} = \# \left(S_1^{+}(x) \setminus S_1^{\mathrm{cyc}}(x)\right) = \sum_{y \in (S_1^{\mathrm{cyc}}(x))^c} \mathbb{1}_{\{x \sim y, D_x \le D_y\}}.$$

By Proposition 6.2.5, there exists a constant C_ν such that with ν-high probability, we have

$$D_x^{+} \le C_\nu + D_x^{\mathrm{nc}+}.$$

We thus only have to bound $D_x^{\mathrm{nc}+}$. For each $y \in S_1(x)$, we introduce the random variable

$$\hat{D}_y = \# \left(S_1(y) \setminus (\{x\} \cup S_1(x))\right) = \sum_{z \in [N] \setminus \{x\}} \mathbb{1}_{\{y \sim z, z \notin S_1(x)\}}.$$

Conditionally to $S_1(x)$, the variables $\hat{D}_y, y \in S_1(x)$ are independent. Furthermore, for $y \in S_1(x)$ with $k \in \mathbf{N}^*$, we have the inclusion of events

$$\{y \notin S_1^{\mathrm{cyc}}(x), D_x \le D_y\} \subset \{y \notin S_1^{\mathrm{cyc}}(x), D_x \le \hat{D}_y + 1\} \subset \{D_x \le \hat{D}_y + 1\}.$$

The union bound and Markov's inequality imply

$$\mathbb{P}\left(D_x^{\mathrm{nc}+} \ge k \mid S_1(x)\right) \le \frac{1}{k!} \sum_{\substack{y_1, \dots, y_k \in S_1(x) \\ \text{distinct}}} \mathbb{P}\left(\forall i, D_x \le \hat{D}_{y_i} + 1 \mid S_1(x)\right) \le \frac{1}{k!} \sum_{\substack{y_1, \dots, y_k \in S_1(x) \\ \text{distinct}}} \prod_{i=1}^{k} \frac{d_{y_i} + 1}{D_x}.$$

Now, notice that as the y_i in the sum are distinct, we have $D_x \geq k$. In particular, if we take $k = \lceil \frac{3\nu}{2(1-\delta)} \frac{\ln N}{\ln\ln N} \rceil$, we can apply the second part of Lemma 6.2.3. With ν-high probability,

$$\mathbb{P}\left(D_x^{\mathrm{nc}+} \geq k \mid S_1(x)\right) \leq \frac{\left(\frac{8(1-\delta)}{3} \ln\ln N\right)^k}{k!} \sum_{\substack{y_1,\dots,y_k \in S_1(x)\\ \text{distinct}}} \prod_{i=1}^{k} \frac{d_{y_i}+1}{d_x}.$$

Finally, taking the expectation of this quantity and using (6.2), we get

$$\begin{aligned}\mathbb{P}\left(D_x^{\mathrm{nc}+} \geq k\right) &\leq \frac{\left(\frac{8(1-\delta)}{3} \ln\ln N\right)^k}{k!} \sum_{\substack{y_1,\dots,y_k\\ \text{distinct}}} \prod_{i=1}^{k} \frac{w_{y_i}(w_{y_i}+1)}{m_1 N} + \mathcal{O}(N^{-\nu}) \\ &= \frac{1}{k!}\left(\frac{8(1-\delta)}{3}\frac{m_2+m_1}{m_1}\ln\ln N\right)^k + \mathcal{O}(N^{-\nu}).\end{aligned}$$

With $k = \lceil \frac{3\nu}{2(1-\delta)} \frac{\ln N}{\ln\ln N} \rceil$, we get from Stirling's asymptotic equivalent that

$$\begin{aligned}\mathbb{P}\left(D_x^{\mathrm{nc}+} \geq k\right) &\leq \frac{1}{\sqrt{2\pi k}} \exp\left(-\frac{3\nu}{2(1-\delta)}\frac{\ln N}{\ln\ln N}\left(\ln k - \ln\left(2\frac{m_2}{m_1}\ln\ln N\right) + \mathcal{O}(1)\right)\right) + \mathcal{O}(N^{-\nu}) \\ &= \frac{1}{\sqrt{2\pi k}} \exp\left(-3\nu \ln N / 2(1+o(1))\right) + \mathcal{O}(N^{-\nu}),\end{aligned}$$

as soon as $(m_2/m_1) < (\ln N)^\delta$, which is satisfied by Hypothesis 6.1.3. □

6.2.4 The pruning procedure

Recall that we fixed $r \geq 6$. In Section 6.3, we do not work in the graph G, but rather in a pruned graph G^{p}. To construct it, we proceed in two steps. Firstly, we do a first pruning procedure to remove some cycles in balls of radius r around all vertices. Then, we remove the down-up path, see Definition 6.2.7. More precisely,

1. for each $x \in [N]$, and each $y \in S_1^{\mathrm{cyc}}(x)$, we remove from G the edge $\{x, y\}$, and then
2. for each $x \in [N]$, and each $y \in S_1^{\mathrm{du}}(x) \setminus S_1^{\mathrm{cyc}}(x)$, we remove from G the edge $\{x, y\}$.

We denote the graph obtained after the first step by G^{nc}, and the resulting graph by G^{p}. We indicate by the superscript p that the adjacency matrix, degrees, spheres, balls, ... correspond to the pruned graph G^{p}. For instance, D_x^{p} is the degree of the vertex x in G^{p}. The event that the edge $\{x, y\}$ is present is denoted by $\{x \overset{\mathrm{p}}{\sim} y\}$.

Theorem 6.2.10. *Let $\nu > 0$. The graph G^{p} satisfies the following properties.*

1. *With ν-high probability, for all $x \in [N]$, $D_x - D_x^{\mathrm{p}} \leq \xi/2$.*
2. *There are no down-up paths in the graph G^{p}.*
3. *The graph G^{p} is a forest.*

Proof. Let $x \in [N]$. During the pruning procedure, we remove at most $D_x^+ + \#\left(S_1^{\mathrm{du}}(x) \setminus S_1^{\mathrm{cyc}}(x)\right) + \# S_1^{\mathrm{cyc}}(x)$ edges around x. Proposition 6.2.5, and Lemmata 6.2.9 and 6.2.8 imply claim 1. The two other claims are consequence of the construction of G^{p}. Let us detail the third one. Assume that there is a simple loop in G^{p} composed of the vertices $(\gamma_0, \gamma_1, \dots, \gamma_k = \gamma_0)$ with $\{\gamma_{i-1}, \gamma_i\} \in E$ for all $i \in [k]$. Then there is a vertex, say γ_0, which is minimal for the total order $\prec$. Then $\gamma_{k-1} \succ \gamma_k = \gamma_0 \prec \gamma_1$, and either $(\gamma_{k-1}, \gamma_0, \gamma_1)$ or $(\gamma_1, \gamma_0, \gamma_{k-1})$ is a down-up path. As there is no such path in G^{p}, there are no cycle in G^{p}. □

6.2.5 Estimate of $\|A - A^{\mathrm{p}}\|$

We now give estimates for the error we make when working with the adjacency matrix of the pruned graph A^{p} rather than the adjacency matrix of the original graph A.

Proposition 6.2.11. *Let $\nu > 0$. There exists a constant $C_\nu > 0$ such that with ν-high probability,*

$$\|A - A^{\mathrm{p}}\| \le C_\nu \frac{\ln N}{\ln \ln N}.$$

This Proposition relies on the following Lemma.

Lemma 6.2.12. *Let $y \in [N]$ and $\nu > 0$. With ν-high probability,*

$$\mathcal{P}_y^{(1)} = \sum_{\substack{x\in[N] \\ y' \notin S_1^{\mathrm{cyc}}(x)\cup\{y\}}} \mathbb{1}_{\{y\notin S_1^{\mathrm{cyc}}(x), \exists z', y\sim x\sim y'\sim z', y,y'\prec x\prec z'\}} < \frac{6\nu}{(1-3\delta)(1-2\delta)} \frac{\ln N}{\ln \ln N}.$$

Proof of Proposition 6.2.11. The matrix $A - A^{\mathrm{p}}$ is the adjacency matrix of the graph made of the edges we removed during the pruning. Consider the adjacency matrix $A^{\mathrm{n.c.}}$ of the graph G^{nc} obtained after the first pruning (removing some cycles near the vertices of $\mathcal{V}$). The maximum degree of a vertex in the graph described by $A - A^{\mathrm{n.c.}}$ is bounded by a constant C_ν, with ν-high probability. It implies that $\|A - A^{\mathrm{n.c.}}\| \le C_\nu$, with ν-high probability. Thus, it suffices to bound $\|A^{\mathrm{n.c.}} - A^{\mathrm{p}}\|$.

Let us introduce for convenience the matrix

$$\tilde{A} = \sum_x 1_x 1_{S_1^{\mathrm{du}}(x)\setminus S_1^{\mathrm{cyc}}(x)} = \sum_{x,y} \mathbb{1}_{\{y\notin S_1^{\mathrm{cyc}}(x), \exists z, x\sim y\sim z, y\prec x\prec z\}} 1_x 1_y^*,$$

we then have

$$A^{\mathrm{p}} - A^{\mathrm{n.c.}} = \tilde{A} + \tilde{A}^*.$$

We only have to bound the operator norm of $\tilde{A}$. We have

$$\|\tilde{A}\|^2 = \max_{\|u\|=1} u^* \tilde{A}^* \tilde{A} u = \max_{\|u\|=1} \sum_{x\in[N]} \sum_{y,y'\in S_1^{\mathrm{du}}(x)\setminus S_1^{\mathrm{cyc}}(x)} u_y u_{y'}.$$

If $y = y'$ in the sum above, we have the contribution

$$\sum_{x\in[N]} \sum_{y\in S_1^{\mathrm{du}}(x)\setminus S_1^{\mathrm{cyc}}(x)} u_y^2 \le \sum_{y\in[N]} u_y^2 D_y^+ \le \frac{3(\nu+1)}{2(1-\delta)} \frac{\ln N}{\ln\ln N},$$

by Lemma 6.2.9. For the remaining terms, Young's inequality then implies

$$\|\tilde{A}\|^2 \le \max_{\|u\|=1} \sum_y u_y^2 \sum_{\substack{x\in[N] \\ y'\notin S_1^{\mathrm{cyc}}(x)\cup\{y\}}} \mathbb{1}_{\{\exists z,z', z\sim y\sim x\sim y'\sim z', y,y'\prec x\prec z,z'\}} + \frac{3(\nu+1)}{2(1-\delta)} \frac{\ln N}{\ln\ln N}.$$

Lemma 6.2.12 allows us to conclude. □

Proof of Lemma 6.2.12. We proceed as in the proof of Lemma 6.2.8. The union bound yields

$$\begin{aligned}
\mathbb{P}\left(\mathcal{P}_y^{(1)} \ge k\right) &= \mathbb{P}\left(\exists I \subset \left\{ (x,y') \in [N]\times([N]\setminus\{y\}) \colon \begin{matrix} y,y' \notin S_1^{\mathrm{cyc}}(x), \\ \exists z', y\sim x\sim y'\sim z', \\ y,y'\prec x\prec z' \end{matrix} \right\}, |I| = k\right) \\
&\le \frac{1}{k!} \sum_{\substack{x_1,\dots,x_k \\ y_1',\dots,y_k' \text{ distinct}}} \mathbb{P}\left(\forall i, y, y_i' \notin S_1^{\mathrm{cyc}}(x_i), \exists z_i' \ne x, y\sim x_i\sim y_i'\sim z_i', D_y, D_{y_i'} \le D_{x_i} \le D_{z_i'}\right) \\
&\le \frac{1}{k!} \sum_{\substack{x_1,\dots,x_k \\ y_1',\dots,y_k' \text{ distinct}}} \mathbb{P}\left(\forall i, y, y_i' \notin S_1^{\mathrm{cyc}}(x_i), \exists z_i' \ne x_i, y\sim x_i\sim y_i'\sim z_i', D_y \le D_{z_i'}\right).
\end{aligned}$$

Notice that in the second line the y_i' can be taken distinct as otherwise, if $(x_i, y_i') \neq (x_j, y_j')$ with $y_i' = y_j'$ then $y \sim x_i \sim y_i' = y_j' \sim x_j \sim y$ is a cycle. This contradicts $y \notin S_1^{\mathrm{cyc}}(x)$.

Assuming the vertices $\{x_i, y_i'\}$ are fixed, we introduce for all $z' \in [N]$

$$\hat{D}_{z'} = \sum_{v \notin \{x_j, y_j', j \in [k]\} \cup \{y\}} \mathbb{1}_{\{z' \sim v\}}.$$

Using this notation, we have

$$\mathbb{P}\left(\mathcal{P}_y^{(1)} \geq k\right) \leq \frac{1}{k!} \sum_{\substack{x_1, \dots, x_k \\ y_1', \dots, y_k' \neq y, \text{ distinct}}} \mathbb{P}\left(\forall i, \exists z_i' \neq x_i, y \sim x_i \sim y_i' \sim z_i', D_y \leq \hat{D}_{z_i'} + 1\right).$$

Conditionally to $S_1(y)$ the variables $\mathbb{1}_{\{x_i \sim y_i'\}}, \mathbb{1}_{\{\exists z_i', y_i' \sim z_i', D_y \leq \hat{D}_{z_i'}+1\}}, i \in [k]$ are independent, with

$$\mathbb{P}\left(\exists z_i' \neq x_i', y_i' \sim z_i', D_y \leq \hat{D}_{z_i'} + 1 \mid S_1(y)\right) \leq \sum_{z' \neq x_i'} p_{y_i', z'} \frac{d_{z'} + 1}{D_y} \leq \frac{w_{y_i'}}{D_y} \frac{m_2 + m_1}{m_1}.$$

We used Markov's inequality in the first inequality, and (6.1), (6.3), and (6.2) for the last one. Then,

$$\mathbb{P}\left(\mathcal{P}_y^{(1)} \geq k\right) \leq \frac{1}{k!} \sum_{\substack{x_1, \dots, x_k \\ y_1', \dots, y_k' \neq y, \text{ distinct}}} \mathbb{E}\left[\prod_{i=1}^{k} \mathbb{1}_{\{y \sim x_i\}} \frac{w_{x_i} w_{y_i'}^2}{m_1 N D_y} \frac{m_2 + m_1}{m_1}\right].$$

Similarly, (6.1) and (6.3) give

$$\mathbb{P}\left(\mathcal{P}_y^{(1)} \geq k\right) \leq \frac{1}{k!} \left(\frac{m_2 + m_1}{m_1}\right)^{2k} \sum_{x_1, \dots, x_k} \mathbb{E}\left(\prod_{i=1}^{k} \mathbb{1}_{\{y \sim x_i\}} \frac{w_{x_i}}{D_y}\right).$$

To sum on distinct vertices x_i, we transform the sum as follows:

$$\begin{aligned}
\mathbb{P}\left(\mathcal{P}_y^{(1)} \geq k\right) &\leq \frac{1}{k!} \left(\frac{m_2 + m_1}{m_1}\right)^{2k} \sum_{l=1}^{k} \binom{k}{l} \sum_{\substack{x_1, \dots, x_l \\ \text{distinct}}} \mathbb{E}\left(\prod_{i=1}^{l} \mathbb{1}_{\{y \sim x_i\}} \frac{w_{x_i} \mathbb{1}_{\{D_y \geq l\}}}{D_y}\right) \\
&\leq \frac{1}{k!} \left(\frac{m_2}{m_1}\right)^{2k} \sum_{l=1}^{k} \binom{k}{l} \sum_{\substack{x_1, \dots, x_l \\ \text{distinct}}} \prod_{i=1}^{l} \frac{w_{x_i}^2}{m_1 N} \frac{w_y \mathbb{1}_{\{D_y \geq l\}}}{D_y}.
\end{aligned}$$

Now, we use again (6.1), to obtain

$$\mathbb{P}\left(\mathcal{P}_y^{(1)} \geq k\right) \leq \frac{1}{k!} \left(\frac{m_2}{m_1}\right)^{2k} \sum_{l=1}^{k} \binom{k}{l} \left(\frac{m_2}{m_1} \frac{w_y \mathbb{1}_{\{D_y \geq l\}}}{D_y}\right)^{l}.$$

Now, let $p \geq 1$ and take $k = \lceil \frac{2p\nu}{1-3\delta} \frac{\ln N}{\ln \ln N} \rceil$. We have

$$\sum_{l=1}^{k} \binom{k}{l} \left(\frac{m_2}{m_1} \frac{w_y \mathbb{1}_{\{D_y \geq l\}}}{D_y}\right)^{l} \leq \sum_{l=1}^{\lfloor k/p \rfloor} \binom{k}{l} \left(\frac{m_2}{m_1} 4\nu \ln N\right)^{l} + \sum_{l=\lfloor k/p \rfloor + 1}^{k} \binom{k}{l} \left(\frac{m_2}{m_1} 2 \ln \ln N (1 + o(1))\right)^{l},$$

where we used the first and second claim of Lemma 6.2.3 in the first and second terms respectively. Now, notice that

$$\begin{aligned}
\sum_{l=1}^{\lfloor k/p \rfloor} \binom{k}{l} \left(\frac{m_2}{m_1} 4\nu \ln N\right)^{l} &\leq \frac{k!}{[(\lfloor k/p \rfloor)!]^2} \sum_{l=1}^{\lfloor k/p \rfloor} \binom{\lfloor k/p \rfloor}{l} \left(\frac{m_2}{m_1} 4\nu \ln N\right)^{l} \\
&= \frac{k!}{(\lfloor k/p \rfloor)!(\lfloor (p-2)k/p \rfloor)!} \left(1 + 4\nu \frac{m_2}{m_1} \ln N\right)^{\lfloor k/p \rfloor}.
\end{aligned}$$

Stirling's approximation then yields

$$\begin{aligned}
&\frac{\left(\frac{m_2+m_1}{m_1}\right)^{2k}\left(1+4\nu\frac{m_2}{m_1}\ln N\right)^{\lfloor k/p\rfloor}}{(\lfloor k/p\rfloor)!(\lfloor (p-2)k/p\rfloor)!}\\
&\qquad\le \exp\left(-\frac{k}{p}\left((p-1)\ln k-\ln\ln N-(2p+1)\ln\ln\frac{m_2}{m_1}\right)(1+o(1))\right)\\
&\qquad= \mathcal{O}\big(N^{-\nu}\big),
\end{aligned}$$

as soon as $(p-2)-(2p+1)\delta \geq 1-3\delta$, i.e. as soon as $p \geq \frac{3-2\delta}{1-2\delta}$. We thus have

$$\mathbb{P}\left(\mathcal{P}_y^{(1)} \geq k\right) \leq \frac{1}{k!}\left(\frac{m_2}{m_1}\right)^{2k}\left(1+3\ln\ln N\frac{m_2}{m_1}\right)^{k}+\mathcal{O}\big(N^{-\nu}\big).$$

Using again Stirling's asymptotics yields the results. □

6.3 Finding the eigenvectors

In this Section, we construct a family of orthonormal vectors that will be close to eigenvectors of A. The main intuition to show that the eigenvectors associated to the greatest eigenvalues are (semi-)localized, is that these eigenvectors are probably close to the vectors

$$v_\sigma(x)=\frac{1}{\sqrt{2}}\left(1_x+\frac{\sigma}{\sqrt{D_x^{\mathrm{p}-}}}1_{S_1^{\mathrm{p}-}(x)}\right),$$

where $x \in [N]$ and $\sigma \in \{\pm 1\}$. This particular choice is motivated by the fact that such vectors are eigenvectors of the adjacency matrix of a star of degree $D_x^{\mathrm{p}-}$, i.e. a tree with one vertex connected to $D_x^{\mathrm{p}-}$ leaves.

Notice that these vectors are normalized, but in the pruned graph G^{p}, we have (recall Definition 6.1.11)

$$v_\rho(x)^* v_\sigma(y)=\frac{\delta_{x,y}}{2}(1+\rho\sigma)+\frac{\mathbb{1}_{\{x\sim y\}}}{2}\left(\frac{\rho\mathbb{1}_{\{x\prec y\}}}{\sqrt{D_x^{\mathrm{p}-}}}+\frac{\sigma\mathbb{1}_{\{x\succ y\}}}{\sqrt{D_y^{\mathrm{p}-}}}\right),$$

that is, we only have $v_-(x)^* v_+(x)=0$ for all $x\in[N]$, but in general two such vectors will not be orthogonal if $x\sim y$.

Remark 6.3.1. In the pruned graph G^{p}, for each vertex $x \in \mathcal{V}$, there is at most one vertex in $\mathcal{V}\cap S_1^+(x)$. Indeed, if it were not the case, i.e. if there existed two distinct vertices $y,z\in\mathcal{V}\cap S_1^+(x)$, then one of $y\sim x\sim z$ or $z\sim x\sim y$ would be a down-up path. There are no such path in the pruned graph.

This remark allows the following definition.

Definition 6.3.2. *Let $x\in\mathcal{V}$. The unique element $y\in\mathcal{V}\cap S_1^+(x)$, if it exists, is called the* ***parent*** *of x, and denoted by $\hat{x}$. Conversely, the* ***children*** *of x are the vertices in $S_1^{\mathrm{p}-}(x)$. The elements of $S_1^{\mathrm{p}-}(\hat{x})$ are called the* ***siblings*** *of x. The set of siblings of a vertex x is denoted by $\mathrm{Sib}(x)$ and is empty if x has no parent. The set of smaller siblings is*

$$\#\,\mathrm{Sib}^-(x)=\mathrm{Sib}(x)\cap\{y\in[N]\colon y\prec x\}.$$

Convention 6.3.3. We use the following conventions.

1. $\frac{1_\emptyset}{0}=0$,
2. any term where the symbol $\hat{x}$ appears is 0 if the vertex x has no parent.

The orthonormal family we shall use is defined in the following Proposition.

Proposition 6.3.4. *For all $x \in \mathcal{V}$, $\sigma \in \{\pm 1\}$, define*

$$Z_x = 2 + 2/\#\operatorname{Sib}^-(x),$$

and

$$u_\sigma(x) = \frac{1}{\sqrt{Z_x}}\left(1_x + \sigma\frac{1_{S_1^{\mathrm{p}-}(x)}}{\sqrt{D_x^{\mathrm{p}-}}} - \frac{1_{\operatorname{Sib}^-(x)}}{\#\operatorname{Sib}^-(x)}\right).$$

The family $(u_\sigma(x))_{x\in\mathcal{V},\sigma\in\{\pm1\}}$ is orthonormal.

Proposition 6.4.1 will imply that the operator

$$\sum_{\substack{x\in\mathcal{V}\\ \sigma\in\{\pm1\}}} \sigma\sqrt{D_x^-}\,u_\sigma(x)u_\sigma(x)^*, \tag{6.5}$$

is a good approximation of A^{p} restricted to the eigenvectors of its greatest eigenvalues. The matrix given by

$$\sum_{\substack{x\in\mathcal{V}\\ \sigma\in\{\pm1\}}} \sigma\sqrt{D_x^-}\,v_\sigma(x)v_\sigma(x)^*, \tag{6.6}$$

is also the adjacency matrix of the pruned graph restricted to the neighborhood of the vertices in $\mathcal{V}$. We now show in Proposition 6.3.5 that the matrices (6.5) and (6.6) are similar.

Proposition 6.3.5. *Let $\nu > 0$. There exists a constant $C_\nu > 0$ such that with ν-high probability,*

$$\left\|\sum_{\substack{x\in\mathcal{V}_\nu\\ \sigma\in\{\pm1\}}} \sigma\sqrt{D_x^{\mathrm{p}-}}\left(u_\sigma(x)u_\sigma(x)^* - v_\sigma(x)v_\sigma(x)^*\right)\right\| \le C_\nu\sqrt{\frac{\ln N}{\ln\ln N}}.$$

We now turn to the proofs. Introduce the families of vectors $(V_0(x))_{x\in\mathcal{V}}$, $(V_1(x))_{x\in\mathcal{V}}$, defined by

$$V_0(x) = 1_x, \quad \text{and} \quad V_1(x) = \frac{1_{S_1^{\mathrm{p}-}(x)}}{\sqrt{D_x^{\mathrm{p}-}}}.$$

In particular, we have $v_\sigma(x) = (V_0(x) + \sigma V_1(x))/\sqrt{2}$.

The family of vectors we shall consider is the family $(U_1(x), U_0(x))_{x\in\mathcal{V}}$ obtained after applying the Gram-Schmidt orthonormalization procedure on $(V_1(x), V_0(x))_{x\in\mathcal{V}}$, starting with the vectors of $(V_1(x))$, and ordering the vectors of $(V_0(x))$ decreasingly according to the order $\prec$.

Working in the pruned graph makes this procedure simpler. Theorem 6.2.10 implies that the family $(V_1(x))_{x\in\mathcal{V}}$ is orthonormal. Indeed, if $x \neq y$, then $V_1(x)^*V_1(y)$ is nonzero if and only if there is a down-up path between $x \in \mathcal{V}$ and $y \in \mathcal{V}$, and in the pruned graph there are no such paths. Thus, we set $U_1(x) = V_1(x)$ for all $x \in \mathcal{V}$.

The resulting vectors $U_0(x)$ for $x \in \mathcal{V}$, are defined by

$$\begin{cases} \tilde{U}_0(x) &= V_0(x) - \sum_{y\in\mathcal{V}}\left(V_0(x)^*U_1(y)\right)U_1(y) - \sum_{\substack{y\in\mathcal{V}\\ y\succ x}}\left(V_0(x)^*U_0(y)\right)U_0(y) \\ U_0(x) &= \frac{\tilde{U}_0(x)}{\|\tilde{U}_0(x)\|}. \end{cases}$$

We then have

$$\begin{aligned}\tilde{U}_0(x) &= 1_x - \sum_{y\in\mathcal{V}} \frac{1_x^* 1_{S_1^{\mathrm{p}-}(y)}}{D_y^{\mathrm{p}-}} 1_{S_1^{\mathrm{p}-}(y)} - \sum_{\substack{y\in\mathcal{V}\\ y\succ x}} \left(1_x^* U_0(y)\right) U_0(y) \\ &= 1_x - \sum_{y\in\mathcal{V}} \frac{\mathbb{1}_{\{x\sim y, x\prec y\}}}{D_y^{\mathrm{p}-}} 1_{S_1^{\mathrm{p}-}(y)} - \sum_{\substack{y\in\mathcal{V}\\ y\succ x}} \left(1_x^* U_0(y)\right) U_0(y) \\ &= 1_x - \frac{1}{D_{\hat{x}}^{\mathrm{p}-}} 1_{S_1^{\mathrm{p}-}(\hat{x})} - \sum_{y\succ x} \left(1_x^* U_0(y)\right) U_0(y).\end{aligned} \tag{6.7}$$

Lemma 6.3.6. *For all $x \in \mathcal{V}$, the vector $U_0(x)$ is supported on $\{x\} \cup \mathrm{Sib}(x)$.*

Proof. We proceed by induction. We first notice that for all $x \in \mathcal{V}$ which have no parent, we have that $\tilde{U}_0(x) = 1_x$.

Then, considering (6.7), we see that

$$1_x - \frac{1}{D_{\hat{x}}^{\mathrm{p}-}} 1_{S_1^{\mathrm{p}-}(\hat{x})}$$

is supported on $\{x\} \cup S_1^{\mathrm{p}-}(\hat{x})$.

By the induction hypothesis $1_x^* U_0(y)$ with $y \succ x$, is non-zero only if $x \in S_1^{\mathrm{p}-}(\hat{y})$, i.e. x and y are siblings. Thus,

$$\sum_{\substack{y\in\mathcal{V}\\ y\succ x}} \left(1_x^* U_0(y)\right) U_0(y)$$

is supported on $\{x\} \cup S_1^{-}(\hat{x})$. Indeed, the siblings of the siblings y of x are the siblings of x. ☐

This Lemma serves as heuristics to prove Proposition 6.3.4.

Proof of Proposition 6.3.4. We first look for the expression of $U_0(x)$, for all $x \in \mathcal{V}$. Lemma 6.3.6 implies that the vector $U_0(x)$ is of the form

$$U_0(x) = a_x 1_x + \sum_{y\in S_1^{\mathrm{p}-}(\hat{x})} b_x(y) 1_y.$$

As the family $(U_1(x), U_0(x))$ is orthonormal, we have for all $x, y \in \mathcal{V}$,

$$0 = U_0(x)^* U_1(y) = \delta_{y,\hat{x}} \left(\frac{a_x}{\sqrt{D_y^{\mathrm{p}-}}} + \sum_{z\in S_1^{\mathrm{p}-}(\hat{x})} \frac{b_x(z)}{\sqrt{D_y^{\mathrm{p}-}}} \right). \tag{6.8}$$

Furthermore, for all $x, y \in \mathcal{V}$,

$$\delta_{x,y} = U_0(x)^* U_0(y) = a_x^2 \delta_{x,y} + \delta_{\hat{x},\hat{y}} \left(a_x b_y(x) + a_y b_x(y)\right) + \delta_{\hat{x},\hat{y}} \sum_{z\in S_1^{\mathrm{p}-}(\hat{x})} b_x(z) b_y(z). \tag{6.9}$$

This implies than if $x = y$

$$1 = a_x^2 + 2a_x b_x(x) + \sum_{z\in S_1^{\mathrm{p}-}(\hat{x})} b_x(z)^2,$$

and if $x \neq y$ with $\hat{x} = \hat{y}$,

$$0 = a_x b_y(x) + a_y b_x(y) + \sum_{z\in S_1^{\mathrm{p}-}(\hat{x})} b_x(z) b_y(z)$$

Consider the particular choice

$$a_x = \frac{1}{\sqrt{Z_x 2}} \quad \text{and} \quad b_x(y) = -\frac{\mathbb{1}_{\{y \prec x\}}}{\sqrt{Z_x/2 \# \operatorname{Sib}^-(x)}},$$

if x has a parent and $b_y(x) = 0$ otherwise. With this choice of coefficients, (6.8) and (6.9) are satisfied. The family

$$(\hat{U}_1(x), \hat{U}_0(x))_{x\in\mathcal{V}} = \left(\frac{1_{S_1^-(x)}}{\sqrt{D_x^{\mathrm{p}-}}}, a_x 1_x + \sum_{y \in S_1^{\mathrm{p}-}(\hat{x})} b_x(y) 1_y \right)_{x\in\mathcal{V}}$$

is then orthonormal. Note however that we do not claim that it is the family obtained from the orthonormalization of $(V_1(x), V_0(x))$. This does not matter: defining $u_\sigma(x) = \frac{\hat{U}_0(x)+\sigma\hat{U}_1(x)}{\sqrt{2}}$ yields an orthonormal family. ☐

We now turn to the proof of Proposition 6.3.5. We first need to prove Lemmata 6.3.7 and 6.3.8 to bound the number of vertices in a ball around a vertex x, and of siblings of a vertex x. Lemma 6.3.7 will also be used in the proof of Proposition 6.4.1.

Lemma 6.3.7. *Let $x \in [N]$ and $\nu > 0$. With ν-high probability, if $x \in \mathcal{V}$ then*

$$\frac{1}{D_x^{\mathrm{p}}} \sum_{y \in S_1^{\mathrm{p}-}(x)} D_y^{\mathrm{p}-} \le \frac{4\nu}{1-\delta} \frac{\ln N}{\ln\ln N} + 1.$$

Lemma 6.3.8. *Let $\nu > 0$. With ν-high probability, for all $x \in \mathcal{V}_\nu$ that has a parent, we have*

$$\# \operatorname{Sib}^-(x) \ge \frac{1}{2} D_{\hat{x}}^{\mathrm{p}}.$$

Proof of Lemma 6.3.7. We introduce the notation

$$\mathcal{P}_x^{(2)} = \sum_{y \in S_1^-(x) \setminus S_1^{\mathrm{cyc}}(x)} (D_y - 1) = \sum_{y,z \neq x} \mathbb{1}_{\{y \notin S_1^{\mathrm{cyc}}(x), x\sim y\sim z, y \prec x\}}.$$

It suffices to bound $\mathcal{P}_x^{(2)}$ to bound $\sum_{y\in S_1^{\mathrm{p}-}(x)} D_y^{\mathrm{p}-}$. Indeed: if $\mathcal{P}_x^{(2)} \le D_x \left(\frac{2\nu}{1-\delta} \frac{\ln N}{\ln\ln N} \right)$ with ν-high probability, we have by Theorem 6.2.10

$$\frac{1}{D_x^{\mathrm{p}}} \sum_{y \in S_1^{\mathrm{p}-}(x)} D_y^{\mathrm{p}-} \le \frac{\mathcal{P}_x^{(2)} + D_x}{D_x} \frac{D_x}{D_x^{\mathrm{p}}} \le \left(\frac{2\nu}{1-\delta} \frac{\ln N}{\ln\ln N} + 1 \right) \frac{D_x}{D_x - \xi/2} \le 2 \left(\frac{2\nu}{1-\delta} \frac{\ln N}{\ln\ln N} + 1 \right).$$

Which give the result. We now prove the required bound on $\mathcal{P}_x^{(2)}$.

Introduce the set

$$V_2(x) := \{(y,z) \in ([N] \setminus S_1^{\mathrm{cyc}}(x)) \times ([N] \setminus \{x\}) \colon x \sim y \sim z, y \prec x\}.$$

For all integer $k \ge 1$, we have

$$\begin{aligned} \mathbb{P}\left(\mathcal{P}_x^{(2)} \ge k \mid S_1(x) \right) &\le \mathbb{P}\left(\exists I \subset V_2(x), \#I = k \mid S_1(x) \right) \\ &\le \frac{1}{k!} \sum_{(y_i,z_i)} \mathbb{P}\left(\forall i \in [k], x \sim y_i \sim z_i, D_{y_i} \le D_x \mid S_1(x) \right), \end{aligned}$$

where the sum is on families of k pairs of vertices (y_i, z_i), such that the $z_1, \dots, z_k$ are distinct. Then, by (6.2), we have

$$\mathbb{P}\left(\mathcal{P}_x^{(2)} \geq k \mid S_1(x)\right) \leq \frac{1}{k!} \sum_{(y_i, z_i)} \mathbb{P}\left(\forall i \in [k], x \sim y_i, D_{y_i} - \mathbb{1}_{\{y_i \sim z_i\}} + 1 \leq D_x \mid S_1(x)\right) \prod_{i=1}^{k} \left(\frac{w_{y_i} w_{z_i}}{m_1 N}\right).$$

Notice that $D_{y_i} - \mathbb{1}_{\{y_i \sim z_i\}} + 1 \geq D_{y_i}$. Using (6.1), we get

$$\mathbb{P}\left(\mathcal{P}_x^{(2)} \geq k \mid S_1(x)\right) \leq \frac{1}{k!} \sum_{(y_i)} \mathbb{P}\left(\forall i \in [k], x \sim y_i, D_{y_i} \leq D_x \mid S_1(x)\right) \prod_{i=1}^{k} w_{y_i}.$$

Now, we rewrite the sum on (y_i) in terms of a sum on families of distinct vertices $(\tilde{y}_i)$:

$$\mathbb{P}\left(\mathcal{P}_x^{(2)} \geq k \mid S_1(x)\right) \leq \frac{1}{k!} \sum_{l=1}^{k} \sum_{\substack{k_1 + \dots k_l = k \\ k_i \geq 1}} \sum_{(\tilde{y}_i)} \mathbb{E}\left[\prod_{i=1}^{l} \mathbb{1}_{\{x \sim y_i, D_{y_i} \leq D_x\}} \mid S_1(x)\right] \prod_{i=1}^{l} w_{\tilde{y}_i}^{k_i},$$

where the sum is on families of l distinct vertices $(\tilde{y}_i)$.

We now condition on $D_x = M$ for some $M \geq \lfloor \frac{\ln N}{\ln \ln N} \rfloor$. By Lemma 6.2.3, we get that with ν-high probability, for all i,

$$\mathbb{1}_{\{D_{y_i} \leq D_x\}} w_{y_i}^{k_i} \leq \left(\frac{D_x w_{y_i}}{D_{y_i} \vee \left(\frac{\ln N}{\ln \ln N}\right)}\right)^{k_i - 1} w_{y_i} \leq (4M(\nu + 1) \ln \ln N)^{k_i - 1} w_{\tilde{y}_i}.$$

Thus, we get using $\sum_{k_1 + \dots + k_l = k} 1 = \binom{k}{l}$,

$$\begin{aligned} &\mathbb{P}\left(\mathcal{P}_x^{(2)} \geq k \mid D_x = M\right) \\ &\leq \frac{1}{k!} \sum_{l=1}^{k} \binom{k}{l} \sum_{(\tilde{y}_i)} \mathbb{E}\left[\prod_{i=1}^{l} \mathbb{1}_{\{x \sim y_i\}} w_{\tilde{y}_i} \left(4(\nu + 1) M \ln \ln N\right)^{k-l} \mid D_x = M\right] + \mathcal{O}\big(N^{-\nu}\big). \end{aligned} \tag{6.10}$$

We must compute the probability

$$\mathbb{P}\left(\forall i \in [l], x \sim \tilde{y}_i \mid D_x = M\right) = \frac{\mathbb{P}\left(D_x = M \mid \forall i \in [l], x \sim \tilde{y}_i\right)}{\mathbb{P}\left(D_x = M\right)} \prod_{i=1}^{l} p_{x\tilde{y}_i}.$$

Estimating $\mathbb{P}\left(D_x = M \mid \forall i \in [l], x \sim \tilde{y}_i\right)$ is done by considering the random variable

$$\hat{D}_x = \sum_{y \notin \{x, \tilde{y}_i; i \in [l]\}} \mathbb{1}_{\{x \sim y\}}.$$

Sums of Bernoulli random variables such as $\hat{D}_x$ have probability densities asymptotically equal to those of Poisson random variables, as we now explain. Let us assume that $M = o(\sqrt{N})$. This can be done without loss of generality as it is implied with very high probability by Hypothesis 6.1.2 and Lemma 6.1.10. Similarly, for $\hat{D}_x$ we have

$$\mathbb{P}\left(\hat{D}_x = M - l\right) = \sum_{\substack{I \subset [N] \setminus \{x\} \\ \# I = M - l}} \prod_{y \in I} p_{xy} \prod_{z \notin I} (1 - p_{xz}),$$

with

$$\prod_{z \notin I} (1 - p_{xz}) = \exp\left(-\sum_{z \notin I} p_{xz}\big(1 + \mathcal{O}\big(N^{-2\epsilon}\big)\big)\right) = \exp\left(-d_x + o(N^{-\epsilon})\right),$$

as $\#I \leq M = o(\sqrt{N})$ and thus using Hypothesis 6.1.3: $\sum_{z\in I} \frac{w_x w_z}{m_1 N} = \mathcal{O}(N^{-\epsilon})$. We get

$$\mathbb{P}\left(\hat{D}_x = M - l\right) = \frac{1}{(M-l)!} \sum_{\substack{y_1,\ldots,y_{M-l} \\ \text{distinct}}} \prod_{i=1}^{M-l} \frac{w_x w_{y_i}}{m_1 N} \mathrm{e}^{-d_x} \left(1 + \mathcal{O}(N^{-\epsilon})\right)$$
$$= \frac{1}{(M-l)!} w_x^{M-l} \mathrm{e}^{-d_x} \left(1 + \mathcal{O}(N^{-\epsilon})\right).$$

Similarly, we have

$$\mathbb{P}\left(D_x = M\right) = \frac{1}{M!} w_x^{M} \mathrm{e}^{-d_x} \left(1 + \mathcal{O}(N^{-\epsilon})\right).$$

Hence, we have

$$\mathbb{P}\left(\forall i \in [l], x \sim \tilde{y}_i \mid D_x = M\right) = \frac{M! w_x^{-l}}{(M-l)!} \prod_{i=1}^{l} p_{x\tilde{y}_i} \left(1 + \mathcal{O}(N^{-\epsilon})\right).$$

Plugging this into (6.10), we have

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{P}_x^{(2)} \geq k \mid D_x = M\right) \\
&\leq \frac{1}{k!} \sum_{l=1}^{k} \binom{k}{l} \sum_{(\tilde{y}_i)} \left(\prod_{i=1}^{l} \frac{w_x w_{\tilde{y}_i}^2}{m_1 N}\right) w_x^{-l} M^l \left(4M\nu \ln\ln N\right)^{k-l} + \mathcal{O}(N^{-\nu}) \\
&= \frac{M^k}{k!} \sum_{l=1}^{k} \binom{k}{l} \sum_{(\tilde{y}_i)} \left(\frac{m_2}{m_1}\right)^l \left(4\nu \ln\ln N\right)^{k-l} + \mathcal{O}(N^{-\nu}) \\
&\leq \frac{M^k}{k!} \left(\frac{m_2}{m_1} + 4\nu \ln\ln N\right)^k + \mathcal{O}(N^{-\nu}).
\end{aligned}$$

We now choose $k = \lfloor 2\nu M \frac{\ln N}{\ln\ln N} \rfloor$. Stirling's asymptotics imply

$$\mathbb{P}\left(\mathcal{P}_x^{(2)} \geq k \mid D_x = M\right) \leq \exp(-k\ln(k/M)(1+o(1))) + \mathcal{O}(N^{-\nu}) = \mathcal{O}(N^{-M\nu}).$$

Finally, we have

$$\mathbb{P}\left(x \in \mathcal{V}, \mathcal{P}_x^{(2)} \geq 2\nu D_x \frac{\ln N}{\ln\ln N} \mid x \in \mathcal{V}_\nu\right) = \mathcal{O}(N^{-\nu}).$$

□

Proof of Lemma 6.3.8. To show this, we shall rather show that the random variable

$$D_x^{\mathrm{nc},\mathcal{V}} = \#\left(S_1(x) \cap \mathcal{V}_\nu \setminus S_1^{\mathrm{cyc}}(x)\right) = \sum_{y \neq x} \mathbb{1}\left(C_\nu \frac{\ln N}{\ln\ln N}\right)_{\{x\sim y, y \notin S_1^{\mathrm{cyc}}(x), \xi < D_y\}},$$

is bounded by $D_x/4$, with $\nu+1$-high probability.

Fix $M > \xi$. We have by the union bound

$$\mathbb{P}\left(D_x^{\mathrm{nc},\mathcal{V}} \geq k \mid D_x = M\right) \leq \frac{1}{k!} \sum_{\substack{y_1,\ldots,y_k \\ \text{distinct}}} \mathbb{P}\left(\forall i \in [k], x \sim y_i, y_i \notin S_1^{\mathrm{cyc}}(x), \xi < D_{y_i} \mid D_x = M\right).$$

Using the fact that the vertices y_i are not in $S_1^{\mathrm{cyc}}(x)$, D_y can be replaced by

$$\hat{D}_{y_i} = \#\left(S_1(y_i) \setminus \{x, y_1, \ldots, y_k\}\right) = \sum_{z \notin \{y_j, x\}} \mathbb{1}_{\{y_i \sim z\}}.$$

We have

$$\mathbb{P}\left(D_x^{\mathrm{nc},\mathcal{V}} \geq k \mid D_x = M\right) \leq \frac{1}{k!} \sum_{\substack{y_1,\dots,y_k \\ \text{distinct}}} \mathbb{P}\left(\forall i, x \sim y_i \mid D_x = M\right) \mathbb{P}\left(\xi < \hat{D}_{y_i} + 1\right).$$

Markov's inequality and (6.2) allows us to conclude that

$$\begin{aligned}\mathbb{P}\left(D_x^{\mathrm{nc},\mathcal{V}} \geq k \mid D_x = M\right) &\leq \frac{1}{k!} \sum_{\substack{y_1,\dots,y_k \\ \text{distinct}}} \mathbb{P}\left(\forall i, x \sim y_i \mid D_x = M\right) \prod_{i=1}^{k} \frac{w_{y_i}}{\xi} \\ &\leq \frac{1}{k!} \sum_{\substack{y_1,\dots,y_k \\ \text{distinct}}} \prod_{i=1}^{k} \frac{w_x w_{y_i}^2}{m_1 N \xi} \frac{\mathbb{P}\left(D_x = M \mid \forall i, x \sim y_i\right)}{\mathbb{P}\left(D_x = M\right)}.\end{aligned}$$

The conditional probability in the right-hand side can be shown to be close to the probability density of a Poisson variable, as in the proof of Lemma 6.3.7:

$$\frac{\mathbb{P}\left(D_x = M \mid \forall i, x \sim y_i\right)}{\mathbb{P}\left(D_x = M\right)} \leq \frac{M!}{(M-k)!} w_x^{-k}.$$

Using (6.1), we get

$$\mathbb{P}\left(D_x^{\mathrm{nc},\mathcal{V}} \geq k \mid D_x = M\right) \leq \frac{M!}{k!(M-k)!} \left(\frac{m_2}{\xi m_1}\right)^k.$$

Taking $k = \lceil M/4 \rceil$, Stirling's asymptotics imply

$$\mathbb{P}\left(D_x^{\mathrm{nc},\mathcal{V}} \geq k \mid D_x = M\right) \leq \exp\left(-\frac{M}{4} \ln \frac{\xi m_1}{m_2} (1 + o(1))\right).$$

Note that Hypothesis 6.1.3 gives that

$$\ln \frac{\xi m_1}{m_2} = \left(\frac{2}{3} \ln \ln N\right) (1 + o(1)),$$

and thus, as $\xi \geq 8\nu \ln N / \ln \ln N$,

$$\mathbb{P}\left(D_x^{\mathrm{nc},\mathcal{V}} \geq k \mid D_x > \xi\right) = \mathcal{O}\left(N^{-\nu}\right).$$

Thus, if $x \in \mathcal{V}_\nu$,

$$D_x^{\mathrm{nc},\mathcal{V}} \leq \frac{1}{4} D_x.$$

with ν-high probability.

As $x \in \mathcal{V}_\nu$, the neighbors of $\hat{x}$ are in $\mathrm{Sib}^-(x)$, or in $\{y \in S_1(\hat{x}) \cap \mathcal{V}_\nu\}$. Thus,

$$\# \mathrm{Sib}^-(x) + D_{\hat{x}}^{\mathrm{nc},\mathcal{V}} \geq D_{\hat{x}}^{\mathrm{p}}.$$

The previous result on $D_{\hat{x}}^{\mathrm{nc},\mathcal{V}}$ implies that with ν-high probability,

$$\# \mathrm{Sib}^-(x) \geq D_{\hat{x}}^{\mathrm{p}} - D_{\hat{x}}^{\mathrm{nc},\mathcal{V}} \geq D_{\hat{x}}^{\mathrm{p}} - \frac{1}{4} D_{\hat{x}} \geq \frac{1}{2} D_{\hat{x}}^{\mathrm{p}},$$

where we used that when $x \in \mathcal{V}_\nu$, by Theorem 6.2.10, $D_x - D_x^{\mathrm{p}} \leq \xi/2$ which implies $D_{\hat{x}}/2 \leq D_{\hat{x}} - \xi/2 \leq D_{\hat{x}}^{\mathrm{p}}$. □

Proof of Proposition 6.3.5. Firstly, notice that

$$\sum_{\sigma\in\{\pm 1\}} \sigma\sqrt{D_x^{\mathrm{p}-}}u_\sigma(x)u_\sigma(x)^* = \frac{2}{Z_x}\left(\sum_{\sigma\in\{\pm 1\}} \sigma\sqrt{D_x^{\mathrm{p}-}}v_\sigma(x)v_\sigma(x)^* - \frac{1_{\mathrm{Sib}^-(x)}1^*_{S_1^-(x)} + 1_{S_1^-(x)}1^*_{\mathrm{Sib}^-(x)}}{\#\,\mathrm{Sib}^-(x)}\right),$$

because of the cancellations occurring when summing contribution of $\sigma = +1$ and $\sigma = -1$.

We consider first

$$\sum_{x\in\mathcal{V}_\nu}\left(\frac{2}{Z_x}-1\right)\sum_{\sigma\in\{\pm 1\}}\sigma\sqrt{D_x^{\mathrm{p}-}}v_\sigma(x)v_\sigma(x)^*,$$

whose operator norm satisfies

$$\begin{aligned}
&\|\sum_{x\in\mathcal{V}_\nu}\left(\frac{2}{Z_x}-1\right)\sum_{\sigma\in\{\pm 1\}}\sigma\sqrt{D_x^{\mathrm{p}-}}v_\sigma(x)v_\sigma(x)^*\|^2 = \|\sum_{x\in\mathcal{V}_\nu}\frac{1_x1^*_{S_1^-(x)} + 1_{S_1^-(x)}1^*_x}{\#\,\mathrm{Sib}^-(x)}\|^2(1+o(1))\\
&\le \max_{\|u\|=1}\sum_{x,x'\in\mathcal{V}_\nu}\frac{1}{\#\,\mathrm{Sib}^-(x)^2}\left(\delta_{x,x'}(1^*_{S_1^-(x)}u)^2 + \delta_{x,x'}D_x^-u_x^2 + \mathbb{1}_{\{x\sim x', x'\prec x\}}u_x(1^*_{S_1^-(x')}u)\right)(1+o(1)).
\end{aligned}$$

Lemma 6.3.8 implies that with ν-high probability, for all $x\in\mathcal{V}_\nu$,

$$\frac{1}{2}D_x^- \le \#\,\mathrm{Sib}^-(x).$$

Together with Young's Lemma and the fact that $1_{S_1^-(x)}$ form an orthogonal family in the pruned graph, it implies

$$\left\|\sum_{x\in\mathcal{V}_\nu}\left(\frac{2}{Z_x}-1\right)\sum_{\sigma\in\{\pm 1\}}\sigma\sqrt{D_x^{\mathrm{p}-}}v_\sigma(x)v_\sigma(x)^*\right\|^2 = \mathcal{O}(1/\xi).$$

Consider now the operator norm

$$\begin{aligned}
\left\|\sum_{x\in\mathcal{V}_\nu}\frac{1_{\mathrm{Sib}^-(x)}1^*_{S_1^{\mathrm{p}-}(x)}}{\#\,\mathrm{Sib}^-(x)}\right\|^2 &\le \max_{\|u\|=1}\sum_{x,x'\in\mathcal{V}_\nu}\delta_{x,x'}\frac{D_x^{\mathrm{p}-}\left(1^*_{\mathrm{Sib}^-(x)}u\right)\left(1^*_{\mathrm{Sib}^-(x')}u\right)}{\left(\#\,\mathrm{Sib}^-(x)\right)^2}\\
&+ \max_{\|u\|=1}2\sum_{x,x'\in\mathcal{V}_\nu}\delta_{\hat{x},\hat{x}'}\mathbb{1}_{\mathrm{Sib}^-(x)}(x')\frac{\left(1^*_{S_1^{\mathrm{p}-}(x)}u\right)\left(1^*_{S_1^{\mathrm{p}-}(x')}u\right)}{\#\,\mathrm{Sib}^-(x)}.
\end{aligned}$$

The first term is bounded by

$$\begin{aligned}
\sum_{x,x'\in\mathcal{V}_\nu}\delta_{x,x'}\frac{D_x^{\mathrm{p}-}\left(1^*_{\mathrm{Sib}^-(x)}u\right)\left(1^*_{\mathrm{Sib}^-(x')}u\right)}{\left(\#\,\mathrm{Sib}^-(x)\right)^2} &= \sum_{x\in\mathcal{V}_\nu}\frac{D_x^{\mathrm{p}-}}{\#\,\mathrm{Sib}^-(x)^2}\sum_{y,y'\in\mathrm{Sib}^-(x)}u_yu_{y'}\\
&\le 2\sum_{x\in\mathcal{V}_\nu}\frac{D_x^{\mathrm{p}-}}{D_{\hat{x}}^{\mathrm{p}-}}\sum_{y\in\mathrm{Sib}^-(x)}u_y^2,
\end{aligned}$$

with ν-high probability. We used Lemma 6.3.8 and Young's inequality. This can be bounded as follows:

$$\sum_{x,x'\in\mathcal{V}_\nu}\delta_{x,x'}\frac{D_x^{\mathrm{p}-}\left(1^*_{\mathrm{Sib}^-(x)}u\right)\left(1^*_{\mathrm{Sib}^-(x')}u\right)}{\left(\#\,\mathrm{Sib}^-(x)\right)^2} \le 2\sum_y u_y^2\sum_{x\in\mathcal{V}_\nu\cap S_1^-(\hat{y})}\frac{D_x^{\mathrm{p}-}}{D_{\hat{y}}^{\mathrm{p}-}}.$$

Lemma 6.3.7 then gives that this is of order $\ln N/\ln\ln N$ with very high probability.

Similarly, using Young's inequality, Lemma 6.3.8 and then Lemma 6.3.7 we see that the second term is bounded by

$$\begin{aligned}\sum_{x,x'\in\mathcal{V}_\nu}\delta_{\hat{x},\hat{x}'}\mathbb{1}_{\mathrm{Sib}^-(x)}(x')\frac{\left(1^*_{S_1^{\mathrm{p}-}(x)}u\right)\left(1^*_{S_1^{\mathrm{p}-}(x')}u\right)}{\#\,\mathrm{Sib}^-(x)} &\leq 2\sum_{x\in\mathcal{V}_\nu}\left(\frac{1^*_{S_1^{\mathrm{p}-}(x)}u}{\sqrt{D_x^{\mathrm{p}-}}}\right)^2\sum_{x'\in S_1^-(\hat{x})\cap\mathcal{V}_\nu}\frac{D_{x'}^{\mathrm{p}-}}{D_{\hat{x}}^{\mathrm{p}-}}\\ &\leq 2\sum_{x\in\mathcal{V}_\nu}\left(\frac{1^*_{S_1^{\mathrm{p}-}(x)}u}{\sqrt{D_x^{\mathrm{p}-}}}\right)^2 C_\nu\frac{\ln N}{\ln\ln N},\end{aligned}$$

with ν-high probability for some constant $C_\nu > 0$. The fact that $(V_1(x))_{x\in\mathcal{V}_\nu}$ is an orthonormal family allow us to conclude that

$$\left\|\sum_{x\in\mathcal{V}_\nu}\frac{1_{\mathrm{Sib}^-(x)}1^*_{S_1^{\mathrm{p}-}(x)}+1_{S_1^{\mathrm{p}-}(x)}1^*_{\mathrm{Sib}^-(x)}}{\#\,\mathrm{Sib}^-(x)}\right\|\leq 2\left\|\sum_{x\in\mathcal{V}_\nu}\frac{1_{\mathrm{Sib}^-(x)}1^*_{S_1^{\mathrm{p}-}(x)}}{\#\,\mathrm{Sib}^-(x)}\right\| = \mathcal{O}\left(\frac{\ln N}{\ln\ln N}\right).$$

Putting together the two bounds, we get the result. □

6.4 The spectral gap and the semi-localization phenomenon

In this Section, we prove Theorem 6.1.9. To do so, we construct a block-diagonal approximation $\hat{A}$ of A, whose eigenvectors associated to the biggest eigenvalues are the $u_\sigma(x)$ for $\sigma\in\{\pm1\}$ and $x\in\mathcal{V}_\nu$. A spectral gap property is the proved for $\hat{A}$, and can be transferred to a spectral gap property for A.

6.4.1 The block-diagonal approximation

We introduce some notation. It is convenient to consider the orthogonal projections

$$\Pi^{\mathrm{p}} = \sum_{\substack{x\in\mathcal{V}_\nu\\ \sigma\in\{\pm1\}}} u_\sigma(x)u_\sigma(x)^* \quad\text{and}\quad \overline{\Pi}^{\mathrm{p}} = \mathrm{Id}-\Pi^{\mathrm{p}}, \tag{6.11}$$

and the block-diagonal approximation of A^{p},

$$\hat{A} = \sum_{\substack{x\in\mathcal{V}_\nu\\ \sigma\in\{\pm1\}}}\sigma\sqrt{D_x^{\mathrm{p}-}}u_\sigma(x)u_\sigma(x)^* + \overline{\Pi}A^{\mathrm{p}}\overline{\Pi}. \tag{6.12}$$

Proposition 6.4.1. *Let $\nu>0$. There exists $C_\nu>0$ such that with ν-high probability,*

$$\|\hat{A}-A^{\mathrm{p}}\|\leq C_\nu\sqrt{\frac{\ln N}{\ln\ln N}}\quad\textit{and}\quad\|\hat{A}-A\|\leq C_\nu\sqrt{\frac{\ln N}{\ln\ln N}}.$$

To prove this Proposition we use the following Lemma.

Lemma 6.4.2. *Define for all $x\in\mathcal{V},\sigma\in\{\pm1\}$,*

$$\delta_\sigma(x) = A^{\mathrm{p}}u_\sigma(x)-\sigma\sqrt{D_x^{\mathrm{p}-}}u_\sigma(x).$$

This vector can also be expressed as

$$\delta_\sigma(x) = \frac{1}{\sqrt{Z_x}}\left(\sigma\sum_{y\in S_1^{\mathrm{p}-}(x)}\frac{1_{S_1^{\mathrm{p}}(y)\setminus\{x\}}}{\sqrt{D_x^{\mathrm{p}-}}}-\sum_{y\in S_1^{\mathrm{p}-}(\hat{x})}\frac{\mathbb{1}_{\{y\prec x\}}}{\#\,\mathrm{Sib}^-(x)}1_{S_1^{\mathrm{p}-}(y)}+\sigma\sqrt{D_x^{\mathrm{p}-}}\frac{1_{\mathrm{Sib}^-(x)}}{\#\,\mathrm{Sib}^-(x)}\right).$$

Proof of Lemma 6.4.2. We compute $A^{\mathrm{p}} u_\sigma(x)$:

$$A^{\mathrm{p}} u_\sigma(x) = \frac{1}{\sqrt{Z_x}} \left(1_{S_1^{\mathrm{p}}(x)} + \frac{\sigma}{\sqrt{D_x^{\mathrm{p}-}}} (D_x^{\mathrm{p}-} 1_x + \sum_{y \in S_1^{\mathrm{p}-}(x)} 1_{S_1^{\mathrm{p}}(y) \setminus \{x\}}) - \sum_{y \in S_1^{\mathrm{p}-}(\hat{x})} \frac{\mathbb{1}_{\{y \prec x\}}}{\# \operatorname{Sib}^-(x)} 1_{S_1^{\mathrm{p}}(y)} \right)$$

$$= \sigma \sqrt{D_x^-} u_\sigma(x)$$

$$+ \frac{\sigma}{\sqrt{Z_x}} \left(\sum_{y \in S_1^-(x)} \frac{1_{S_1^{\mathrm{p}}(y) \setminus \{x\}}}{\sqrt{D_x^-}} - \sigma \sum_{y \in S_1^{\mathrm{p}-}(\hat{x})} \frac{\mathbb{1}_{\{y \prec x\}}}{\# \operatorname{Sib}^-(x)} 1_{S_1^{\mathrm{p}}(y) \setminus \{\hat{x}\}} + \sqrt{D_x^{\mathrm{p}-}} \frac{1_{\operatorname{Sib}^-(x)}}{\# \operatorname{Sib}^-(x)} \right).$$

The result follows when noticing that in the pruned graph $S_1^{\mathrm{p}}(y) \setminus \{\hat{y}\} = S_1^{\mathrm{p}-}(y)$. □

Proof of Proposition 6.4.1. The triangular inequality yields

$$\|\hat{A} - A\| \le \|\hat{A} - A^{\mathrm{p}}\| + \|A^{\mathrm{p}} - A\|.$$

Proposition 6.2.11 allows us to bound the second part. The first part can be bounded as follows

$$\|\hat{A} - A^{\mathrm{p}}\| \le \| \sum_{\substack{x \in \mathcal{V}_\nu \\ \sigma \in \{\pm 1\}}} \sigma \sqrt{D_x^-} u_\sigma(x) u_\sigma(x)^* - \Pi^{\mathrm{p}} A^{\mathrm{p}} \Pi^{\mathrm{p}} \| + \|\overline{\Pi}^{\mathrm{p}} A^{\mathrm{p}} \Pi^{\mathrm{p}} + \Pi^{\mathrm{p}} A^{\mathrm{p}} \overline{\Pi}^{\mathrm{p}}\|.$$

Recall that in Lemma 6.4.2, we defined the vector

$$\delta_\sigma(x) = A^{\mathrm{p}} u_\sigma(x) - \sigma \sqrt{D_x^{\mathrm{p}-}} u_\sigma(x) \quad \text{for all } x \in \mathcal{V}, \sigma \in \{\pm 1\}.$$

We have

$$A^{\mathrm{p}} \Pi^{\mathrm{p}} = \sum_{\substack{x \in \mathcal{V}_\nu \\ \sigma \in \{\pm 1\}}} A^{\mathrm{p}} u_\sigma(x) u_\sigma(x)^* = \sum_{\substack{x \in \mathcal{V}_\nu \\ \sigma \in \{\pm 1\}}} \sigma \sqrt{D_x^{\mathrm{p}-}} u_\sigma(x) u_\sigma(x)^* + B,$$

where $B = \sum_{\substack{x \in \mathcal{V}_\nu \\ \sigma \in \{\pm 1\}}} \delta_\sigma(x) u_\sigma(x)^*$. We now bound the operator norm of B. Using the expression of $\delta_\sigma(x)$ from Lemma 6.4.2, we write the operator as a sum $B = B_1 + B_2 + B_3 + B_4$, with

$$B_1 = \sum_{x \in \mathcal{V}_\nu} \frac{2}{Z_x} \sum_{y \in S_1^-(x)} \frac{1_{S_1^{\mathrm{p}-}(y)} 1^*_{S_1^{\mathrm{p}-}(x)}}{D_x^{\mathrm{p}-}}$$

$$B_2 = \sum_{x \in \mathcal{V}_\nu} \frac{2}{Z_x} \sum_{\substack{y \in S_1^{\mathrm{p}-}(\hat{x}) \\ y \prec x}} \frac{1_y 1^*_{S_1^{\mathrm{p}-}(x)}}{\# \operatorname{Sib}^-(x)}$$

$$B_3 = -\sum_{x \in \mathcal{V}_\nu} \frac{2}{Z_x} \sum_{\substack{y \in S_1^{\mathrm{p}-}(\hat{x}) \\ y \prec x}} \frac{1_{S_1^{\mathrm{p}-}(y)} 1^*_x}{\# \operatorname{Sib}^-(x)}$$

$$B_4 = \sum_{x \in \mathcal{V}_\nu} \frac{2}{Z_x} \sum_{\substack{y,z \in S_1^{\mathrm{p}-}(\hat{x}) \\ y,z \prec x}} \frac{1_{S_1^{\mathrm{p}-}(y)} 1^*_z}{(\# \operatorname{Sib}^-(x))^2}.$$

We now bound the operator norm of each of the four operators. The first one is

$$\|B_1\|^2 = \max_{\|u\|=1} (u^* B_1^* B_1 u) = \max_{\|u\|=1} \sum_{x,x' \in \mathcal{V}_\nu} \frac{4}{Z_x Z_{x'}} \frac{u^* 1_{S_1^{\mathrm{p}-}(x)} 1^*_{S_1^{\mathrm{p}-}(x')} u}{D_x^{\mathrm{p}-} D_{x'}^{\mathrm{p}-}} \sum_{\substack{y \in S_1^{\mathrm{p}-}(x) \\ y' \in S_1^{\mathrm{p}-}(x')}} 1^*_{S_1^{\mathrm{p}-}(y)} 1_{S_1^{\mathrm{p}-}(y')}.$$

The orthogonality of the vectors $(1_{S_1^{\mathrm{p}-}(y)})_{y\in[N]} = (\sqrt{D_y^{\mathrm{p}-}}V_1(y))_{y\in[N]}$ implies that we must take $y = y'$ in the sum above. As $x = \hat{y}$ and $x' = \hat{y}'$ this means that the contributions for $x \neq x'$ vanish. Furthermore, we can assume that u is of the form $u = \sum_{x\in\mathcal{V}_\nu} \alpha_x V_1(x)$. We thus get

$$\|B_1\|^2 = \max_{\sum_{x\in\mathcal{V}_\nu}\alpha_x^2=1} \sum_{x\in\mathcal{V}} \frac{4}{Z_x^2}\frac{\alpha_x^2}{D_x^{\mathrm{p}-}} \sum_{y\in S_1^{\mathrm{p}-}(x)} D_y^{\mathrm{p}-}.$$

Lemma 6.3.7 implies that there exists a constant C_ν such that with ν-high probability

$$\|B_1\|^2 \leq C_\nu \frac{\ln N}{\ln\ln N}.$$

The other bounds are proved similarly, in Section 6.6. Thus, there exists a constant $C'_\nu > 0$ such that with ν-high probability,

$$\|A^{\mathrm{p}}\Pi^{\mathrm{p}}\| \leq C'_\nu \sqrt{\frac{\ln N}{\ln\ln N}}.$$

The result follows from the fact that Π^{p} and $\overline{\Pi}^{\mathrm{p}}$ are orthogonal projections. □

6.4.2 Bounds on $\overline{\Pi} A^{\mathrm{p}} \overline{\Pi}$

The last step before proving Theorem 6.1.9 is to bound the operator norm of $\overline{\Pi} A^{\mathrm{p}} \overline{\Pi}$. To do so, we use the fact that the pruned graph is actually a forest.

Let us consider $\overline{\Pi} A^{\mathrm{p}} \overline{\Pi}$. By definition, see (6.11), we have

$$\begin{aligned}\overline{\Pi} A^{\mathrm{p}} \overline{\Pi} = \hat{A} - \sum_{\substack{x\in\mathcal{V}_\nu\\ \sigma\in\{\pm1\}}} \sigma\sqrt{D_x^-}\, u_\sigma(x)u_\sigma(x)^* &= (A^{\mathrm{p}} - \hat{A}) \\ &- \sum_{\substack{x\in\mathcal{V}_\nu\\ \sigma\in\{\pm1\}}} \sigma\sqrt{D_x^-}\,(u_\sigma(x)u_\sigma(x)^* - v_\sigma(x)v_\sigma(x)^*) \\ &+ (A^{\mathrm{p}} - \sum_{\substack{x\in\mathcal{V}_\nu\\ \sigma\in\{\pm1\}}} \sigma\sqrt{D_x^-}\, v_\sigma(x)v_\sigma(x)^*).\end{aligned}$$

Proposition 6.4.1 implies that the error $\|A^{\mathrm{p}} - \hat{A}^{\mathrm{p}}\|$ is at most of order $\sqrt{\ln N/\ln\ln N}$, and Proposition 6.3.5 that the second part is at most of order $\sqrt{\ln N/\ln\ln N}$. We now prove that the third part is of order $\sqrt{\ln N/\ln\ln N}$.

Lemma 6.4.3. *We have*

$$\|A^{\mathrm{p}} - \sum_{\substack{x\in\mathcal{V}_\nu\\ \sigma\in\{\pm1\}}} \sigma\sqrt{D_x^-}\, v_\sigma(x)v_\sigma(x)^*\| \leq 2\sqrt{\xi}.$$

Proof. The matrix

$$A' = A^{\mathrm{p}} - \sum_{\substack{x\in\mathcal{V}_\nu\\ \sigma\in\{\pm1\}}} \sigma\sqrt{D_x^-}\, v_\sigma(x)v_\sigma(x)^*$$

is the adjacency matrix of the graph $\tilde{G} = G^{\mathrm{p}}|_{\mathcal{V}^c}$. The degree of the vertices of this graph are bounded by ξ, and the graph is actually a forest by Theorem 6.2.10.

Thus, a standard estimate – see for instance [ADK21a, Lemma A.4] – implies that

$$\|A'\| \leq 2\sqrt{\xi}.$$

□

This Lemma yields the following bound on $\|\overline{\Pi}^{\mathrm{p}} A^{\mathrm{p}} \overline{\Pi}^{\mathrm{p}}\|$.

Proposition 6.4.4. *Let $\nu > 0$. There exists a constant $C_\nu > 0$ such that with ν-high probability, we have*

$$\|\overline{\Pi}^{\mathrm{p}} A^{\mathrm{p}} \overline{\Pi}^{\mathrm{p}}\| \leq C_\nu \sqrt{\frac{\ln N}{\ln \ln N}}.$$

6.4.3 Semi-localization

We now prove the main result. Firstly, we introduce some notation. Let $\eta > 0$ and $\lambda > 0$. We define the sets

$$\mathcal{W}^{\mathrm{p}}_{\lambda,\eta} = \{x \in [N] \colon |\sqrt{D^{\mathrm{p}-}_x} - \lambda| \leq \eta\} \quad \text{and} \quad \mathcal{W}_{\lambda,\eta} = \{x \in [N] \colon |\sqrt{D_x} - \lambda| \leq \eta\},$$

and the orthogonal projections

$$\Pi^{\mathrm{p}}_{\lambda,\eta} = \sum_{x \in \mathcal{W}^{\mathrm{p}}_{\lambda,\eta}} u_+(x) u_+(x)^* \quad \text{and} \quad \overline{\Pi}^{\mathrm{p}}_{\lambda,\eta} = 1 - \Pi^{\mathrm{p}}_{\lambda,\eta}.$$

The following semi-localization result, and its proof, are very close to [ADK21a, Theorem 3.4].

Theorem 6.4.5. *Let $\nu > 0$. There exists $c_\nu, C_\nu > 0$ such that with ν-high probability, the following statement is true. For all eigenvalue $|\lambda| > C_\nu \sqrt{\frac{\ln N}{\ln \ln N}}$ with associated normalized eigenvector q, for all $0 < \eta \leq |\lambda|/2$, we have*

$$\sum_{x \in \mathcal{W}^{\mathrm{p}}_{\lambda,\eta}} \langle q, u_+(x) \rangle^2 \geq 1 - \left(\frac{1}{\eta} \frac{c_\nu \sqrt{\ln N}}{\sqrt{\ln \ln N}} \right)^2.$$

Proof. As in [ADK21a, Theorem 3.4], the core of the proof is the spectral gap property of the form

$$\mathrm{Spec}(\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta} A \overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}) \subset \mathbb{R} \setminus [\lambda - \eta, \lambda + \eta].$$

Consider first the block-diagonal approximation $\hat{A}$. The orthogonal projections Π^{p} and $\Pi^{\mathrm{p}}_{\lambda,\eta}$ commute and we have the inclusion property

$$\Pi^{\mathrm{p}} \Pi^{\mathrm{p}}_{\lambda,\eta} = \Pi^{\mathrm{p}}_{\lambda,\eta}.$$

Note that we also have

$$\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta} = \mathrm{Id} - \Pi^{\mathrm{p}} \Pi^{\mathrm{p}}_{\lambda,\eta} = \overline{\Pi}^{\mathrm{p}} + \overline{\Pi}^{\mathrm{p}}_{\lambda,\eta} \Pi^{\mathrm{p}}.$$

These properties allow us to rewrite $\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta} \hat{A} \overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}$ as

$$\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta} \hat{A} \overline{\Pi}^{\mathrm{p}}_{\lambda,\eta} = \overline{\Pi}^{\mathrm{p}}_{\lambda,\eta} \Pi^{\mathrm{p}} \hat{A} \Pi^{\mathrm{p}} \overline{\Pi}^{\mathrm{p}}_{\lambda,\eta} + \overline{\Pi}^{\mathrm{p}} \hat{A} \overline{\Pi}^{\mathrm{p}}. \tag{6.13}$$

The spectral gap property can be shown for $\Pi^{\mathrm{p}}_{\lambda,\eta} \hat{A} \Pi^{\mathrm{p}}_{\lambda,\eta}$ by showing it for each of the two terms in (6.13): they are the two blocks of a block decomposition of the operator. By definition, we immediately have

$$\mathrm{Spec}(\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta} \Pi^{\mathrm{p}} \hat{A} \Pi^{\mathrm{p}} \overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}) \subset \mathbb{R} \setminus [\lambda - \eta, \lambda + \eta].$$

For the second part, Proposition 6.4.4 implies that there exists a constant $C_\nu > 0$ such that with ν-high probability,

$$\mathrm{Spec}(\overline{\Pi}^{\mathrm{p}} \hat{A} \overline{\Pi}^{\mathrm{p}}) \subset [-\frac{C_\nu}{2} \sqrt{\frac{\ln N}{\ln \ln N}}, \frac{C_\nu}{2} \sqrt{\frac{\ln N}{\ln \ln N}}] \subset \mathbb{R} \setminus [\lambda - \eta, \lambda + \eta],$$

as we chose η such that $|\lambda \pm \eta| \geq |\lambda|/2 > C_\nu \sqrt{\ln N / \ln \ln N}/2$.

We have proved

$$\mathrm{Spec}(\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}\hat{A}\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}) \subset \mathbb{R} \setminus [\lambda-\eta, \lambda+\eta].$$

Properties 6.4.1 allows to upgrade this to a spectral gap property for $\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}A\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}$: there exists $c_\nu > 0$ such that with very high probability

$$\mathrm{Spec}(\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}A\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}) \subset \mathbb{R} \setminus [\lambda - (\eta - c_\nu\sqrt{\frac{\ln N}{\ln\ln N}}), \lambda + (\eta - c_\nu\sqrt{\frac{\ln N}{\ln\ln N}})].$$

By convention, if $\eta - c_\nu\sqrt{\frac{\ln N}{\ln\ln N \ln}} > 0$, then the interval is $\emptyset$. Assume that $\eta > c_\nu\sqrt{\frac{\ln N}{\ln\ln N}}$, otherwise the result is vacuous.

We now conclude as follows. Let λ be an eigenvalue associated to a normalized eigenvector q, we have $(A-\lambda)q = 0$. Multiplying by $\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}$ and introducing $\mathrm{Id} = \Pi^{\mathrm{p}}_{\lambda,\eta} + \overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}$, we get

$$\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}(A-\lambda)\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}q + \overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}(A-\lambda)\Pi^{\mathrm{p}}_{\lambda,\eta}q = 0,$$

which simplifies to

$$(\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}A\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta} - \lambda)\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}q = -\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}A\Pi^{\mathrm{p}}_{\lambda,\eta}q.$$

Finally, we have

$$\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}q = -(\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}A\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta} - \lambda)^{-1}\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}A\Pi^{\mathrm{p}}_{\lambda,\eta}q.$$

The spectral gap property and Proposition 6.4.1 imply the bounds

$$\|(\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}A\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta} - \lambda)^{-1}\| \leq \frac{1}{\eta - c_\nu\sqrt{\frac{\ln N}{\ln\ln N}}}$$

$$\|\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}A\Pi^{\mathrm{p}}_{\lambda,\eta}\| \leq \|\hat{A} - A\| \leq c_\nu\sqrt{\frac{\ln N}{\ln\ln N}}.$$

They allow us to deduce

$$\|\overline{\Pi}^{\mathrm{p}}_{\lambda,\eta}q\| \leq \frac{c_\nu\sqrt{\frac{\ln N}{\ln\ln N}}}{\eta - c_\nu\sqrt{\frac{\ln N}{\ln\ln N}}} \wedge 1 \leq \frac{2c_\nu\sqrt{\frac{\ln N}{\ln\ln N}}}{\eta},$$

which is the wanted result. □

Proof of Theorem 6.1.9. Notice that by Theorem 6.2.10, with ν-high probability, $D^{\mathrm{p}}_x \leq D_x - \xi/2$. Thus, assuming $\lambda, \eta \geq \sqrt{\xi}/2$ we get

$$\mathcal{W}^{\mathrm{p}}_{\lambda,\eta-\sqrt{\xi/2}} \subset \mathcal{W}_{\lambda,\eta}. \tag{6.14}$$

Set $\tilde{\eta} = \eta - \sqrt{\frac{\xi}{2}}$. The inclusion (6.14) implies that for all vector u,

$$\|\left(1 - \sum_{x\in\mathcal{W}_{\lambda,\eta}} u_+(x)u_+(x)^*\right)u\| \leq \|\overline{\Pi}^{\mathrm{p}}_{\lambda,\tilde{\eta}}u\|.$$

We have by Theorem 6.4.5 that with ν-high probability,

$$\|\left(1 - \sum_{x\in\mathcal{W}_{\lambda,\eta}} u_+(x)u_+(x)^*\right)q\| \leq \|\overline{\Pi}^{\mathrm{p}}_{\lambda,\tilde{\eta}}q\| \leq \frac{c_\nu\sqrt{\frac{\ln N}{\ln\ln N}}}{\tilde{\eta}} = \frac{c_\nu\sqrt{\frac{\ln N}{\ln\ln N}}}{\eta - \sqrt{\xi/2}},$$

□

6.5 The biggest eigenvalues and the localization phenomenon

We now turn to the biggest eigenvalues. We show that they are close to the square roots of the degrees of some vertices in G. Furthermore, assuming that the average degrees d_x are well separated, we obtain the localization of some eigenvectors around single vertices.

Theorem 6.5.1. *There exists a constant $C_\nu > 0$ such that with ν-high probability, for all $i \in [N]$, if $\lambda_i(A) > C_\nu \sqrt{\frac{\ln N}{\ln \ln N}}$, then*

$$\left|\lambda_i(A) - \sqrt{D_{\pi(i)}}\right| \leq C_\nu \sqrt{\frac{\ln N}{\ln \ln N}}.$$

or if $\lambda_i(A) < -C_\nu \sqrt{\frac{\ln N}{\ln \ln N}}$, then

$$\left|\lambda_i(A) + \sqrt{D_{\pi(i)}}\right| \leq C_\nu \sqrt{\frac{\ln N}{\ln \ln N}}.$$

Proof. This is a consequence of Proposition 6.4.1. There exists $C'_\nu > 0$ such that with ν-high probability

$$|\lambda_i(A) - \lambda_i(\hat{A})| \leq \|A - \hat{A}\| \leq C'_\nu \sqrt{\frac{\ln N}{\ln \ln N}},$$

for all $i \in [N]$.

The i-th eigenvalue of $\hat{A}$ satisfies $\lambda_i(\hat{A}) = \pm\sqrt{D^{\mathrm{p}-}_{\pi(i)}}$ if $\lambda_i(\hat{A}) > \|\overline{\Pi}^{\mathrm{p}} A^{\mathrm{p}} \overline{\Pi}^{\mathrm{p}}\|$. Furthermore, Theorem 6.2.10 implies that with ν-high probability, for all $x \in \mathcal{V}$,

$$|D_x - D^{\mathrm{p}-}_x| \leq \xi/2$$

Hence, there exists a constant C_ν such that with ν-high probability: for all i such that $\lambda_i(\hat{A}) > \|\overline{\Pi}^{\mathrm{p}} A^{\mathrm{p}} \overline{\Pi}^{\mathrm{p}}\|$,

$$|\lambda_i(A) - \sqrt{D_x}| \leq C_\nu \sqrt{\frac{\ln N}{\ln \ln N}}.$$

We have a similar result when $\lambda_i(\hat{A}) < -\|\overline{\Pi}^{\mathrm{p}} A^{\mathrm{p}} \overline{\Pi}^{\mathrm{p}}\|$. ◻

We now consider the phenomenon of localization around a single vertex, in general a stronger result than the semi-localization. According to Theorem 6.4.5, it occurs when $\#\mathcal{W}_{\lambda,\eta} = 1$ for an appropriate pair (λ, η). We fix $\nu > 0$ and $\eta > 0$.

We introduce the set of isolated vertices:

$$\mathcal{V}^*_{\nu,\eta} = \left\{ x \in [N] : \begin{array}{l} \bullet\ \forall y \in [N], y \neq x, |d_x - d_y| \geq \left(4\sqrt{\nu \ln N d_x} + 4\sqrt{\nu \ln N d_y}\right) \vee 16\eta^2 \\ \bullet\ d_x \geq \dfrac{4\nu}{9} \ln N \end{array} \right\}. \tag{6.15}$$

We shall show that the eigenvectors associated to the vertices in $\mathcal{V}^*_{\nu,\eta}$ are localized with ν-high probability.

Theorem 6.5.2. *There exists $C_\nu > 0$ such that with ν-high probability, for all eigenvalue $\lambda > C_\nu \sqrt{\frac{\ln N}{\ln \ln N}}$ of A, with associated eigenvector $\boldsymbol{w}$, and all $\eta \leq \lambda/2$, we have the following property.*

If $\mathcal{W}_{\lambda,\eta} \cap \mathcal{V}^*_{\nu,\eta} \neq \emptyset$ then there exists $x \in \mathcal{V}^*_{\nu,\eta}$ such that

$$\langle \boldsymbol{w}, u_+(x) \rangle^2 \geq 1 - \left(\frac{1}{\eta} \frac{C_\nu \sqrt{\ln N}}{\sqrt{\ln \ln N}} \right)^2.$$

Proof. We shall show that if $\mathcal{W}_{\lambda,\eta} \cap \mathcal{V}^*_{\nu,\eta} \neq \emptyset$, then $\#\mathcal{W}_{\lambda,\eta} = 1$. Theorem 6.5.2 is then a consequence of Theorem 6.4.5.

Let $x \in \mathcal{W}_{\lambda,\eta} \cap \mathcal{V}^*_{\nu,\eta}$ and $y \in \mathcal{W}_{\lambda,\eta}$ with $d_y \geq \frac{4\nu}{9} \ln N$. Note that this second point holds with ν-high probability as soon as we take $C_\nu > 0$ big enough. Lemma 6.1.10 implies that with ν-high probability,

$$d_x - d_y - 2\sqrt{\nu \ln N d_x} - 2\sqrt{\nu \ln N d_y} \leq D_x - D_y \leq d_x - d_y + 2\sqrt{\nu \ln N d_x} + 2\sqrt{\nu \ln N d_y}.$$

Using (6.15), we have

$$d_x - d_y - \frac{1}{2}|d_x - d_y| \leq D_x - D_y \leq d_x - d_y + \frac{1}{2}|d_x - d_y|.$$

Notice that if $D_x > D_y$ then $d_x \geq d_y$ as otherwise we would have

$$D_x - D_y \leq -\frac{1}{2}|d_x - d_y| \leq 0,$$

a contradiction. We can show similarly that if $D_x < D_y$ then $d_x \leq d_y$.

Thus, we have

$$|D_x - D_y| \geq \frac{1}{2}|d_x - d_y|. \tag{6.16}$$

Finally, we have by (6.16) and (6.15),

$$|\sqrt{D_x} - \sqrt{D_y}| \geq \sqrt{\frac{|D_x - D_y|}{2}} \geq \frac{\sqrt{|d_x - d_y|}}{2} \geq 2\eta,$$

and thus either $x = y$ or $y \notin \mathcal{W}_{\lambda,\eta}$. □

Example 6.5.3. Consider Example 6.1.6. The weights are chosen as the quantiles of a law with heavy tail. Let $\alpha > 2$. We set

$$w_i = \left(\frac{N}{i}\right)^{1/\alpha}.$$

In that case, if $i \leq N^{1/(2\alpha+2)}$, we have

$$|w_i - w_{i-1}| \geq \left(\frac{N}{i}\right)^{1/\alpha} \frac{c}{i},$$

for some constant $c > 0$. As the sequence $(|w_{i+1} - w_i|)$ is increasing, we get that $\{1, \ldots, \lfloor N^{1/(2\alpha+2)} \rfloor\} \subset \mathcal{V}^*_{\nu,\eta}$ with $\eta = \frac{\sqrt{\alpha}}{4} N^{1/(4\alpha)}$. On the other hand, Theorem 6.5.1 implies that for $i \leq \lfloor N^{1/(2\alpha+2)} \rfloor$,

$$\left|\lambda_i(A) - \sqrt{D_{\pi(i)}}\right| \ll \eta$$

with ν-high probability, that is for all such i,

$$\pi(i) \in \mathcal{W}_{\lambda_i(A),\eta}.$$

Bennett's inequality shows that $D_i > D_{i+1}$ with ν-high probability for $i \leq \lfloor N^{1/(2\alpha+2)} \rfloor$: thus $\pi(i) = i$ for such i's.

It then follows from Theorem 6.5.2 that with ν-high probability, the eigenvectors corresponding to the $N^{1/(2\alpha+2)}$ first eigenvalues are localized.

6.6 Bounds for the proof of Proposition 6.4.1

Consider the operator

$$B_2 = \sum_{x\in\mathcal{V}} \frac{2}{Z_x} \sum_{\substack{y\in S_1^{\mathrm{p}-}(\hat{x})\\ y\prec x}} \frac{1_{S_1^{\mathrm{p}-}(x)} 1_y^*}{\sqrt{D_x^{\mathrm{p}-}\#\,\mathrm{Sib}^-(x)}}.$$

Its operator norm is

$$\begin{aligned}\|B_2\|^2 = \max_{\|u\|=1} (u^* B_2 B_2^* u) &= \max_{\|u\|=1} \sum_{x,x'\in\mathcal{V}} \frac{4u^* 1_{S_1^{\mathrm{p}-}(x)} 1^*_{S_1^{\mathrm{p}-}(x')} u}{Z_x Z_{x'} \sqrt{D_x^{\mathrm{p}-} D_{x'}^{\mathrm{p}-}\#\,\mathrm{Sib}^-(x)\,\mathrm{Sib}^-_{x'}}} \sum_{\substack{y\in S_1^{\mathrm{p}-}(\hat{x})\\ y\prec x\\ y'\in S_1^{\mathrm{p}-}(\hat{x'})\\ y'\prec x'}} 1_y^* 1_{y'}\\
&= \max_{\|u\|=1} \sum_{x,x'\in\mathcal{V}} \frac{4u^* 1_{S_1^{\mathrm{p}-}(x)} 1^*_{S_1^{\mathrm{p}-}(x')} u}{Z_x Z_{x'} \sqrt{D_x^{\mathrm{p}-} D_{x'}^{\mathrm{p}-}} \sqrt{\#\,\mathrm{Sib}^-(x)\,\mathrm{Sib}^-_{x'}}} \delta_{\hat{x},\hat{x}'}\\
&= \max_{\|u\|=1} \sum_{x\in\mathcal{V}} \frac{4u^* 1_{S_1^{\mathrm{p}-}(x)} 1^*_{S_1^{\mathrm{p}-}(x)} u}{Z_x^2 D_x^{\mathrm{p}-}\#\,\mathrm{Sib}^-(x)}.\end{aligned}$$

We can assume that $u = \sum_{x\in\mathcal{V}} \alpha_x V_1(x)$ and write

$$\|B_2\|^2 \le \max_{\sum_{x\in\mathcal{V}}\alpha_x^2=1} \sum_{x\in\mathcal{V}} \frac{\alpha_x^2}{\#\,\mathrm{Sib}^-(x)} \le 1.$$

We now turn to the operator

$$B_3 = -\sum_{x\in\mathcal{V}} \frac{2}{Z_x^2} \sum_{y\in\mathrm{Sib}^-(x)} \frac{1_{S_1^{\mathrm{p}-}(y)} 1_x^*}{\#\,\mathrm{Sib}^-(x)}.$$

Its operator norm is

$$\|B_3\|^2 = \max_{\|u\|=1} u^* B_3^* B_3 u = \sum_{x,x'\in\mathcal{V}} \frac{4u_x u_{x'}}{Z_x Z_{x'}\#\,\mathrm{Sib}^-(x)\,\mathrm{Sib}^-_{x'}} \sum_{\substack{y\in\mathrm{Sib}^-(x)\\ y'\in\mathrm{Sib}^-(x')}} 1^*_{S_1^{\mathrm{p}-}(y)} 1_{S_1^{\mathrm{p}-}(y')}.$$

The orthogonality of the vectors $(1_{S_1^{\mathrm{p}-}(y)})_{y\in[N]} = (\sqrt{D_y^{\mathrm{p}-}} V_1(y))_{y\in[N]}$ yields

$$\|B_3\|^2 = \sum_{x,x'\in\mathcal{V}} \frac{4\delta_{\hat{x},\hat{x}'} u_x u_{x'}}{Z_x Z_{x'}\#\,\mathrm{Sib}^-(x)\,\mathrm{Sib}^-_{x'}} \sum_{y\in\mathrm{Sib}^-(x)\cap\mathrm{Sib}^-(x')} D_y.$$

We apply Young's inequality to replace $\frac{u_x u_{x'}}{\#\,\mathrm{Sib}^-(x)\,\mathrm{Sib}^-_{x'}}$ by $\frac{u_x^2}{(\#\,\mathrm{Sib}^-(x))^2}$:

$$\|B_3\|^2 \le \sum_{x,x'\in\mathcal{V}} \frac{4\delta_{\hat{x},\hat{x}'} u_x^2}{Z_x Z_{x'} \left(\#\,\mathrm{Sib}^-(x)\right)^2} \sum_{y\in\mathrm{Sib}^-(x)\cap\mathrm{Sib}^-(x')} D_y \le \sum_{x\in\mathcal{V}} \frac{4u_x^2 D_{\hat{x}}}{Z_x \left(\#\,\mathrm{Sib}^-(x)\right)^2} \sum_{y\in\mathrm{Sib}^-(x)} D_y^{\mathrm{p}}.$$

Lemmata 6.3.8 and 6.3.7 give that with ν-high probability

$$\|B_3\|^2 \le \sum_{x,x'\in\mathcal{V}} \frac{4u_x^2}{Z_x} \frac{2\nu}{1-\delta} \frac{\ln N}{\ln\ln N} = \mathcal{O}\left(\frac{\ln N}{\ln\ln N}\right).$$

Finally, consider the operator

$$B_4 = \sum_{x\in\mathcal{V}} \frac{2}{Z_x} \sum_{y,z\in \mathrm{Sib}^-(x)} \frac{1_{S_1^{\mathrm{p}-}(y)} 1_z^*}{(\#\,\mathrm{Sib}^-(x))^2},$$

and define for $x \in \mathcal{V}$,

$$\begin{aligned} u(x) &= \frac{1}{\#\,\mathrm{Sib}^-(x)} 1_{\mathrm{Sib}^-(x)} \\ v(x) &= \frac{1}{\#\,\mathrm{Sib}^-(x)} \sum_{y\in\mathrm{Sib}^-(x)} 1_{S_1^{\mathrm{p}-}(y)}. \end{aligned}$$

Notice that

$$\begin{aligned} B_5 u(x) &= \frac{2}{Z_x(\#\,\mathrm{Sib}^-(x))^3} \sum_{y,z\in\mathrm{Sib}^-(x)} 1_{S_1^{\mathrm{p}-}(y)} = \frac{2}{Z_x(\#\,\mathrm{Sib}^-(x))^2} \sum_{y\in\mathrm{Sib}^-(x)} 1_{S_1^{\mathrm{p}-}(y)} \\ &= \frac{2}{Z_x \#\,\mathrm{Sib}^-(x)} v(x). \end{aligned}$$

Furthermore,

$$v(x)^* B_4 = \frac{2}{Z_x(\#\,\mathrm{Sib}^-(x))^3} \sum_{y,z\in\mathrm{Sib}^-(x)} D_y^{\mathrm{p}-} 1_z^* = \left(\frac{2}{Z_x(\#\,\mathrm{Sib}^-(x))^2} \sum_{y\in\mathrm{Sib}^-(x)} D_y^{\mathrm{p}-} \right) u(x)^*.$$

Thus, Lemmata 6.3.7 and 6.3.8, and the Schur test imply that

$$\|B_4\| = \mathcal{O}(1).$$

6.7 Estimation of the size of $\mathcal{W}_{\lambda,\eta}$ (Proof of Proposition 6.1.8)

Let $\lambda, \eta > 0$ such that $2\sqrt{\frac{\ln N}{\ln\ln N}} \le \eta \le \lambda/2$. We consider the expectation $\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta}\right]$, that we rewrite as

$$\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta}\right] = \sum_{\substack{x\in[N] \\ w_x \le \sqrt{\ln N}}} \mathbb{P}\left((\lambda-\eta)^2 \le D_x \le (\lambda+\eta)^2\right) + \sum_{\substack{x\in[N] \\ w_x > \sqrt{\ln N}}} \mathbb{P}\left((\lambda-\eta)^2 \le D_x \le (\lambda+\eta)^2\right).$$

The first part can be bounded using Bennett's inequality [BLM13, Theorem 2.9]:

$$\mathbb{P}\left(D_x \ge (\lambda-\eta)^2\right) \le \exp\left(-(\lambda-\eta)^2 \ln\left(\frac{((\lambda-\eta)^2)}{d_x} \right) - (\lambda-\eta)^2 + d_x \right) = \mathcal{O}\big(N^{-2}\big),$$

so that

$$\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta}\right] = \sum_{\substack{x\in[N] \\ w_x > \sqrt{\ln N}}} \mathbb{P}\left((\lambda-\eta)^2 \le D_x \le (\lambda+\eta)^2\right) + \mathcal{O}\big(N^{-1}\big).$$

We start be recalling the following Lemma of approximation of the degrees by Poisson variables.

Lemma 6.7.1 (Approximation of degrees by a Poisson variable [vdHof16, Theorem 6.7])**.** *There exists a coupling $(\hat{D}_x, \hat{P}_x)$ of the degree D_x of vertex x and a Poisson variable x with parameter w_x, such that*

$$\mathbb{P}\left(\hat{D}_x \ne \hat{P}_x\right) \le \frac{w_x^2}{m_1 N}\left(1 + 2\frac{m_2}{m_1}\right).$$

This Lemma will be key in estimating $\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta}\right]$ for some $\lambda, \eta > 0$. Indeed, we have

$$\begin{aligned}
\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta}\right] &= \sum_{\substack{x\in[N]\\ w_x>\sqrt{\ln N}}} \mathbb{P}\left((\lambda-\eta)^2 \le D_x \le (\lambda+\eta)^2\right) + \mathcal{O}\left(N^{-1}\right) \\
&\le \sum_{\substack{x\in[N]\\ w_x>\sqrt{\ln N}}} \mathbb{P}\left((\lambda-\eta)^2 \le \hat{P}_x \le (\lambda+\eta)^2\right) + \frac{m_2}{m_1}\left(1+2\frac{m_2}{m_1}\right) + \mathcal{O}\left(N^{-1}\right).
\end{aligned}$$

By Hypothesis 6.1.3, the last term is of order at most $(\ln N)^{2/3}$. The first term can be written in term of incomplete Gamma functions

$$\Gamma(s,x) = \int_x^\infty t^{s-1}\mathrm{e}^{-t}\mathrm{d}t.$$

Indeed, we have

$$\sum_{\substack{x\in[N]\\ w_x>\sqrt{\ln N}}} \mathbb{P}\left((\lambda-\eta)^2 \le \hat{P}_x \le (\lambda+\eta)^2\right) = \sum_{\substack{x\in[N]\\ w_x>\sqrt{\ln N}}} \left(\frac{\Gamma(U_N, w_x)}{\Gamma(U_N)} - \frac{\Gamma(L_N-1, w_x)}{\Gamma(L_N-1)}\right),$$

where we set for convenience $L_N = \lfloor(\lambda-\eta)^2\rfloor$ and $U_N = \lceil(\lambda+\eta)^2\rceil$.

We shall use the two following standard properties of incomplete Gamma function.

Lemma 6.7.2. *Let $s \ge 1$ and $x > 0$. Then, $\Gamma(s) - x^{s-1} \le \Gamma(s,x) \le \Gamma(s)$. Furthermore, $\Gamma(s,x) \sim x^{s-1}\mathrm{e}^{-x}$ as $x \to \infty$.*

Proof. It is immediate that $\Gamma(s,x) \le \Gamma(s)$. For the other bound, we have

$$\Gamma(s,x) = \Gamma(s) - \int_0^x t^{s-1}\mathrm{e}^{-t}\mathrm{d}t \ge \Gamma(s) - x^{s-1}\int_0^x \mathrm{e}^{-t}\mathrm{d}t \ge \Gamma(s) - x^{s-1}.$$

To prove the asymptotic estimate, we remark that

$$\frac{\Gamma(s,x)}{x^{s-1}\mathrm{e}^{-x}} = \int_x^\infty \left(\frac{t}{x}\right)^{s-1}\mathrm{e}^{-(t-x)}\mathrm{d}t = \int_0^\infty \left(\frac{t}{x}+1\right)^{s-1}\mathrm{e}^{-t}\mathrm{d}t.$$

The monotone convergence theorem then implies that the limit of the left-hand term is 1 as $x \to \infty$. □

We kept in the sum only terms x such that $w_x \to \infty$ as $N \to \infty$, hence by Lemma 6.7.2, we have

$$\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta}\right] \le \sum_{\substack{x\in[N]\\ w_x>\sqrt{\ln N}}} \mathrm{e}^{-w_x}\left(\frac{w_x^{U_N-1}}{(U_N-1)!} - \frac{w_x^{L_N-2}}{(L_N-2)!}\right)(1+o(1)) + \mathcal{O}\left((\ln N)^{2\delta}\right).$$

We now consider two cases:

- The weights (w_x) are the $(N+1)$-quantiles of an exponential law as in Example 6.1.5.
- The weights (w_x) are the $(N+1)$-quantiles of law with heavy tails as in Example 6.1.6.

In the exponential case, we then have

$$\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta}\right] \le N\int_0^\infty \alpha\mathrm{e}^{-\alpha t-t}\left(\frac{t^{U_N-1}}{(U_N-1)!} - \frac{t^{L_N-2}}{(L_N-2)!}\right)\mathrm{d}t(1+o(1)) + \mathcal{O}\left((\ln N)^{2\delta}\right).$$

Using that the k-th moment of an exponential law of parameter $\alpha+1$ is $k!/(\alpha+1)^k$ we get

$$\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta}\right] \le N\frac{\alpha}{\alpha+1}\left(\frac{1}{(\alpha+1)^{L_N-2}} - \frac{1}{(\alpha+1)^{U_N-1}}\right)(1+o(1)) + \mathcal{O}\left((\ln N)^{2\delta}\right).$$

That is,

$$\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta}\right] \leq N\frac{\alpha}{(\alpha+1)^{\lfloor(\lambda-\eta)^2\rfloor-1}}(1+o(1)) + \mathcal{O}\Big((\ln N)^{2\delta}\Big). \tag{6.17}$$

In the heavy tail case, we have

$$\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta}\right] \leq N\frac{\alpha^2 t_0^{\alpha+1}}{(\alpha-1)d}\int_{t_0}^{\infty} \mathrm{e}^{-t}\left(\frac{t^{L_N-\alpha-3}}{(L_N-2)!} - \frac{t^{U_N-\alpha-2}}{(U_N-1)!}\right)\mathrm{d}t(1+o(1)) + \mathcal{O}\Big((\ln N)^{2\delta}\Big).$$

Finally, we have

$$\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta}\right] \leq N\frac{\alpha^2 t_0^{\alpha+1}}{(\alpha-1)d}\left(\frac{\Gamma(L_N-\alpha-2,t_0)}{\Gamma(L_N-1)} - \frac{\Gamma(U_N-\alpha-1,t_0)}{\Gamma(U_N)}\right)(1+o(1)) + \mathcal{O}\Big((\ln N)^{2\delta}\Big).$$

Using Lemma 6.7.2, we get

$$\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta}\right] \leq N\frac{\alpha^2 t_0^{\alpha+1}}{(\alpha-1)d}\left(\frac{\Gamma(L_N-\alpha-2)}{\Gamma(L_N-1)} - \frac{\Gamma(U_N-\alpha-1) - t_0^{U_N-\alpha-2}}{\Gamma(U_N)}\right)(1+o(1)) + \mathcal{O}\Big((\ln N)^{2\delta}\Big).$$

Using Stirling's asymptotics we have

$$\begin{aligned}\frac{\Gamma(L_N-\alpha-2)}{\Gamma(L_N-1)} - \frac{\Gamma(U_N-\alpha-1)}{\Gamma(U_N)} &= \left(\left(\frac{L_N}{\mathrm{e}}\right)^{-\alpha-1} - \left(\frac{U_N}{\mathrm{e}}\right)^{-\alpha-1}\right)(1+o(1)) \\ &= \mathrm{e}^{\alpha+1}\left(\frac{1}{(\lambda-\eta)^{2\alpha+2}} - \frac{1}{(\lambda+\eta)^{2\alpha+2}}\right)(1+o(1)).\end{aligned}$$

Finally, we get

$$\mathbb{E}\left[\#\mathcal{W}_{\lambda,\eta}\right] \leq N\frac{\alpha^2 t_0^{\alpha+1}}{(\alpha-1)d}\frac{4(\alpha+1)\mathrm{e}^{\alpha+1}\eta}{\lambda^{2\alpha+3}}(1+o(1)) + \mathcal{O}\Big((\ln N)^{2\delta}\Big). \tag{6.18}$$

We now turn to the proof of Proposition 6.1.8. The proof will use the following variant of Lemma 6.2.9.

Lemma 6.7.3. *For each vertex $x \in [N]$, define*

$$\hat{D}_x^+ = \#\{y \in S_1(x) \colon w_y \geq w_x\} = \sum_{y \neq x} \mathbb{1}_{\{x\sim y, w_x \leq w_y\}}.$$

Let $\nu > 0$. With ν-high probability, we have

$$\hat{D}_x^+ \leq \frac{2\nu}{1-\delta}\frac{\ln N}{\ln\ln N}.$$

Proof of Lemma 6.7.3. Let $k \geq 1$ be an integer. The union bound implies

$$\mathbb{P}\left(\hat{D}_x^+ \geq k\right) \leq \frac{1}{k!}\sum_{\substack{x_1,\ldots,x_k \\ \text{distinct}}} \mathbb{P}\left(\forall i \in [k], x \sim x_i, w_x \leq w_{x_i}\right).$$

By independence and (6.2), we have

$$\mathbb{P}\left(\hat{D}_x^+ \geq k\right) \leq \frac{1}{k!}\sum_{\substack{x_1,\ldots,x_k \\ \text{distinct}}}\prod_{i=1}^{k}\left(\frac{w_x w_{x_i}}{m_1 N}\mathbb{1}_{\{w_x \leq w_{x_i}\}}\right) \leq \frac{1}{k!}\sum_{\substack{x_1,\ldots,x_k \\ \text{distinct}}}\prod_{i=1}^{k}\left(\frac{w_{x_i}^2}{m_1 N}\right),$$

where in the last line, we used that $\mathbb{1}_{\{w_x \leq w_{x_i}\}} \leq w_{x_i}/w_x$. By definition of the second empirical moment, we have

$$\mathbb{P}\left(\hat{D}_x^+ \geq k\right) \leq \frac{1}{k!}\left(\frac{m_2}{m_1}\right)^k(1+o(1)).$$

Taking $k = \lfloor \frac{2\nu}{1-\delta}\frac{\ln N}{\ln\ln N}\rfloor$ allows us to conclude. □

Proof of Proposition 6.1.8. Let $k \geq 1$ be an integer. We use the union bound to write

$$\mathbb{P}\left(\#\mathcal{W}_{\lambda,\eta} \geq k\right) \leq \frac{1}{k!} \sum_{\substack{x_1,\ldots,x_k \\ \text{distinct}}} \mathbb{P}\left(\forall i \in [k], (\lambda-\eta)^2 \leq D_{x_i} \leq (\lambda+\eta)^2\right).$$

Set $\hat{D}_x^- = D_x - \hat{D}_x^+$ for all $x \in [N]$. By Lemma 6.7.3, we have for all $x \in [N]$ that

$$D_x = \hat{D}_x^+ + \hat{D}_x^- \leq \hat{D}_x^- + \frac{2\nu+2}{1-\delta}\frac{\ln N}{\ln\ln N},$$

with ν-high probability. For convenience, write $c_{\nu,N} = \frac{2\nu+2}{1-\delta}\frac{\ln N}{\ln\ln N}$.

Now, notice that the random variables $\hat{D}_x^-$ are independent. It implies

$$\begin{aligned}\mathbb{P}\left(\#\mathcal{W}_{\lambda,\eta} \geq k\right) &\leq \frac{1}{k!} \sum_{\substack{x_1,\ldots,x_k \\ \text{distinct}}} \prod_{i=1}^{k} \mathbb{P}\left((\lambda-\eta)^2 - c_{\nu,N} \leq \hat{D}_{x_i}^- \leq (\lambda+\eta)^2\right) + \mathcal{O}(N^{-\nu}) \\ &\leq \frac{1}{k!}\left(\sum_x \mathbb{P}\left((\lambda-\eta)^2 - c_{\nu,N} \leq \hat{D}_x^- \leq (\lambda+\eta)^2\right)\right)^k + \mathcal{O}(N^{-\nu}).\end{aligned}$$

Notice that we have

$$\begin{aligned}\mathbb{P}\left(\#\mathcal{W}_{\lambda,\eta} \geq k\right) &\leq \frac{1}{k!}\left(\sum_x \mathbb{P}\left((\lambda-\eta)^2 - c_{\nu,N} \leq D_x \leq (\lambda+\eta)^2 + c_{\nu,N}\right)\right)^k + \mathcal{O}(N^{-\nu}) \\ &\leq \frac{\left(\mathbb{E}\#\mathcal{W}_{\lambda,\eta+c_{\nu,N}/2\lambda}\right)^k}{k!} + \mathcal{O}(N^{-\nu}).\end{aligned}$$

Markov inequality gives the crude bound

$$\mathbb{P}\left(\#\mathcal{W}_{\lambda,\eta} \geq k\right) \leq \frac{1}{k!}\left(\sum_x \frac{w_x^2}{\left((\lambda-\eta)^2 - c_{\nu,N}\right)^2}\right)^k = \frac{1}{k!}\left(N\frac{m_2}{\left((\lambda-\eta)^2 - c_{\nu,N}\right)^2}\right)^k.$$

Choosing $k = \lfloor \frac{2m_2}{(\lambda-\eta)^4}N \vee \frac{2\nu \ln N}{\ln\ln N} \rfloor$ gives the result. The result can be improved in our two examples using expressions (6.17) and (6.18), derived above. □

Chapter 7

Future research

We conclude by gathering a few research directions that arise naturally from this Thesis.

7.1 Topological expansion of unitary integrals, and beyond

Our investigation in Chapter 3 was motivated by the study of multi-matrix models. In particular, we imposed condition on the potential so that the measure μ_V^N over the unitary group is a probability measure. We may ask if there is a topological expansion when V is possibly complex. This has been shown by Novak in the case of the HCIZ integral [Nov20]. The same techniques may be used to improve the result of Chapter 3 to any small enough polynomial potential.

A second limitation of our work is that it is restricted to the perturbative regime. Simply showing that the multi-matrix models converge in the large dimension limit is difficult in general – in particular when the potential is not convex.

Beyond the unitary group, we may apply the same program for integrals over the orthogonal group. This had already been studied by Collins, Guionnet, and Maurel-Segala [CGM09] for the leading order. In particular, there are Dyson-Schwinger equations in this case as well. The combinatorial part of this question would be to introduce possibly non-orientable maps of unitary type, and show Tutte-like equations for their generating series. This is similar to what is done when going from perturbation of the GUE measure, to perturbations of the GOE measure.

In the light of Chapter 4, we may even ask what a β-deformed equivalent of these integrals would look like. In the case of the HCIZ integral, this has already been studied [MP22]. To answer this more prospective question in general, one would need to develop a β-deformed Weingarten calculus.

7.2 β-ensembles and maps

In Chapter 4, we proposed a new way of expressing the large N expansion of the cumulant of the β-ensemble. This expansion must coincide with the one of LaCroix, in terms of possibly non-orientable maps. However, we managed to show bijectively the correspondence only for the first two orders. To extend this, a key part would be to extend the many-to-one mapping of Section 4.5 to relate orientable and non-orientable maps of any topology to suitably labelled maps with a prescribed topology and number of local minima.

Understanding the higher order of the expansion could allow the computation of exact formulae for expectations of product of distances in maps of a given topology. This is however dependent on having exact formula to compute cumulants of the β-ensemble. As explained in 4.6 the case of expectations of distances in planar maps is related to the β-ensemble in the $\beta \to \infty$ limit. Furthermore, we showed that the case of planar trees can be obtained, provided we know the asymptotics of power sums of roots of Hermite polynomials.

In another direction, the many-to-one mapping of 4.5 could pave the way to constructing the scaling limits of non-orientable surfaces. Indeed, since the breakthrough of Le Gall [LeG13] and Miermont

[Mie13], the scaling limit of many families of maps have been computed. In particular, higher genus Brownian surfaces have been shown to be the scaling limit of random maps of higher genus, see for instance [Bet10; BM22]. Such constructions often rely on bijections between the studied family of maps, and simpler combinatorial objects. The many-to-one mapping may provide a way to approach the construction of the scaling limit of maps on the projective plane: it relates non-orientable maps on the projective plane, which are usually complicated to work with, and planar maps, which are well-understood. It thus supplements the existing bijections of Chapuy and Dołęga [CD17], and of Bettinelli [Bet22].

7.3 Other Fay-like identites

The identities obtained in Chapter 5 are only valid for hyperelliptic curves. In principle, more general spectral curves can be obtained by considering two-matrix models or linearly coupled chain of matrices.

In such models, exact identities between observables are provided by the Eynard–Mehta formulae [EM98]. Obtaining exact asymptotics in the large dimension limit can be – as mentioned in the last Section – a difficult problem.

In all cases, the ones we treated in Chapter 5 and the prospective situation just described, the interpretation of these formulae remain to be found. There could be a geometric interpretation, akin to the trisecant interpretation of Fay's identity or its role in the Schottky problem, or an interpretation in terms of integrable systems.

7.4 Different regimes for the localization in the GRG model

The results obtained in Chapter 6 depend on two assumptions: an assumption on the behavior of the first and second empirical moments of the sequence of weights, and an assumption on the tails of this distribution. We may wonder how these assumptions can be relaxed. The first one, concerning the moments is not sharp: it does not cover the case of a Erdős-Rényi graph with average degree of order $\ln N$, studied by Alt, Ducatez and Knowles [ADK21a]. We believe however that the hypothesis on the tails is close to optimal. Indeed, when tails become so heavy that the empirical second moments diverges polynomially in N, there may be a number of vertices of very high degree that are all connected together: in this regime, there are pairs of vertices x, y whose weights are such that $w_x w_y$ is of order N. In that case, p_{xy} is of constant order. This is in stark contrast to the case we considered, in which $p_{xy} \simeq \frac{w_x w_y}{m_1 N}$ with $w_x w_y \ll N$. A different study is thus necessary to tackle this very heavy tail case. In particular, different heuristicts are needed to study these large degree vertices: they may form cliques rather than be almost isolated.

Furthermore, a hard question would be to show the delocalization-localization transition in the GRG model. Showing that eigenvectors associated to small eigenvalues are delocalized – without even showing a sharp transition – would be a difficult problem. The derivation of local laws, a key step to prove the delocalization result, would be made more difficult because of the inhomogeneity of the model.

7.5 A new point of view on Dyson-Schwinger equations

In an ongoing work with Alice Guionnet and Slim Kammoun, we study the approach of Kazakov and Zheng [KZ22] to solve experimentally the Dyson-Schwinger equations. They propose to solve the Dyson-Schwinger equations by turning it into a semi-definite programming (SDP) problem, that can efficiently be solved on a computer. However, the equivalence of the SDP problem and of the Dyson-Schwinger equations is not proven in full rigor. Completing the work of Kazakov and Zheng would provide a new point of view on the Dyson-Schwinger equations, by seeing it as an optimization problem.

A key element of the reformulation of Kazakov and Zheng is that, as in the Hamburger problem, a matrix of moments must be positive semi-definite. How this constraint help determine the moments of random matrices remains to be understood.

7.6 Large deviations for the empirical measure of sub-Gaussian Wigner matrices

In an ongoing work with Alice Guionnet and Ella Hiesmayr, we study the perturbation of a GOE matrix G by a subgaussian Wigner matrix R, i.e. the matrix

$$X_\epsilon = (1-\epsilon)G + \epsilon R,$$

for some small $\epsilon > 0$. Our main problem is to prove a large deviation principle for the empirical measure of X_ϵ. A key tool that has been used successfully to derive large deviation principles is based on computation of spherical integrals, see for instance [Hus22; CDG24]. In our case, it turns out that the quantity

$$\lim_{N\to\infty} \frac{1}{N^2} \ln \mathbb{E}\left[\mathrm{e}^{zN\,\mathrm{Tr}\left(RODO^T\right)}\right], \tag{7.1}$$

where D is a deterministic matrix and O is a Haar-distributed orthogonal matrix, is expected to be the most complicated term of the rate function of the large deviation principle. One way to approach the computation of (7.1) is to proceed perturbatively: as in Chapter 3, this quantity is a generating function of cumulants, which may be computed using Weingarten calculus. While we managed to compute these cumulants, showing that the formal expansion does converge to a function that is the quantity (7.1) remains to be done.

Chapter 8

Appendices

8.1 Derivation of the Dyson-Schwinger equations

We derive the Dyson-Schwinger equations for the case of a perturbation of the GUE.

Proof of Lemma 2.6.3. It suffices to show this on monomials

$$X_{q_1}^{k_1} X_{q_2}^{k_2} \cdots X_{q_r}^{k_r},$$

where $q_1, \ldots, q_r \in [l]$, $k_1, \ldots, k_r \in \mathbf{N}$. We then obtain the full result by linearity. We have

$$\begin{aligned}
&\frac{\partial}{\partial (X_q)_{i',j'}} \left(X_{q_1}^{k_1} X_{q_2}^{k_2} \cdots X_{q_l}^{k_l} \right)_{i,j} \\
&\quad = \sum_{p=1}^{l} \sum_{i^{(2)},j^{(2)}=1}^{N} \left(\prod_{p'=1}^{p-1} X_{q_{p'}}^{k_{p'}} \right)_{i,i^{(2)}} \left(\frac{\partial}{\partial (X_q)_{i',j'}} X_{q_p}^{k_p} \right)_{i^{(2)},j^{(2)}} \left(\prod_{p'=p+1}^{r} X_{q_{p'}}^{k_{p'}} \right)_{j^{(2)},j} \\
&\quad = \sum_{p=1}^{l} \sum_{i^{(2)},j^{(2)}=1}^{N} \left(\prod_{p'=1}^{p-1} X_{q_{p'}}^{k_{p'}} \right)_{i,i^{(2)}} \delta_{q,q_p} \left(\partial_{X_q} X_{q_p}^{k_p} \right)_{\substack{i^{(2)},i' \\ j',j^{(2)}}} \left(\prod_{p'=p+1}^{r} X_{q_{p'}}^{k_{p'}} \right)_{j^{(2)},j} \\
&\quad = \left[\sum_{p=1}^{r} \delta_{q,q_p} \left(\prod_{p'=1}^{p-1} X_{q_{p'}}^{k_{p'}} \otimes \mathrm{Id} \right) \times \left(\partial_{X_q} X_q^{k_p} \right) \times \left(\mathrm{Id} \otimes \prod_{p'=p+1}^{r} X_{q_{p'}}^{k_{p'}} \right) \right]_{\substack{i,i' \\ j',j}} \\
&\quad = \left[\partial_{X_q} \left(X_{q_1}^{k_1} X_{q_2}^{k_2} \cdots X_{q_r}^{k_r} \right) \right]_{\substack{i,i' \\ j',j}}.
\end{aligned}$$

In the expression above, Id denotes the identity matrix. Setting $i = j$ in the equality above and summing on i, we obtain the second claim. □

Proof of Theorem 2.6.4. Fix $q \in [m]$. We consider the integral

$$\int_{\mathcal{H}_N^m} \frac{\partial}{\partial (X_q)_{ij}} P_{i',j'} \exp\left(-N \operatorname{Tr} V - \frac{N}{2} \sum_{k=1}^{m} \operatorname{Tr} X_i^2 \right) \mathrm{d}X_1 \cdots \mathrm{d}X_m = 0, \tag{8.1}$$

in which the polynomials P and V are evaluated in the matrices $X_1, \ldots, X_m, A_1, \ldots, A_p$. We shall assume this in the sequel to make notation lighter. We divide by Z_V^N and use the notation $\langle \cdot \rangle_V^N$:

$$\langle f \rangle_V^N = \mu_{2,V}^N [f],$$

where f is any function of the matrices $X_1, \dots, X_m, A_1, \dots, A_l$. We compute the derivatives in (8.1) and obtain

$$\left\langle \left(\frac{\partial}{\partial (X_q)_{ij}} P_{i',j'} \right) \right\rangle_V^N - N \left\langle \left(\frac{\partial}{\partial (X_q)_{ij}} \operatorname{Tr}\left(V + \frac{X_q^2}{2} \right) \right) P_{i',j'} \right\rangle_V^N = 0$$

Lemma 2.6.3 allows us to rewrite this

$$\left\langle (\partial_{x_q} P)_{\substack{i',i \\ j,j'}} \right\rangle_V^N - N \left\langle (\mathcal{D}_{x_q} V + X_q)_{j,i} P_{i',j'} \right\rangle_V^N = 0.$$

Set $i = i'$ and $j = j'$ and sum on both of these indices to obtain the first Dyson-Schwinger equation:

$$\langle \operatorname{Tr} \otimes \operatorname{Tr} (\partial P) \rangle_V^N = N \left\langle \operatorname{Tr} \left((\mathcal{D}_{x_q} V + X_q) P \right) \right\rangle_V^N . \tag{8.2}$$

which can be rewritten in terms of cumulants as

$$\left(\mathcal{W}_{V,1}^N \otimes \mathcal{W}_{V,1}^N + \mathcal{W}_{V,2}^N \right) (\partial P) = N \mathcal{W}_{V,1}^N \left((\mathcal{D}_{x_q} V + X_q) P \right). \tag{8.3}$$

We now consider the potential $\tilde{V} = V + \sum_{k=2}^{l} t_k P_k$. We differentiate (8.3) with respect to $t_2, \dots, t_l$ and then set $t_2 = \dots = t_l = 0$. As we have for any $K = \{i_1, \dots, i_{\#K}\} \subset [l]$, any $l' \geq 1$ and polynomials $Q_1, \dots, Q_{l'}$,

$$\prod_{i \in K} \frac{\partial}{\partial t_i} |_{t_1 = \dots = t_l = 0} \mathcal{W}_{\tilde{V},l'}^N (Q_1, \dots, Q_{l'}) = N^{\#K} \mathcal{W}_{V,l+\#K}^N (Q_1, \dots, Q_{l'}, P_{i_1}, \dots, P_{i_{\#K}}),$$

we obtain the wanted equations. □

8.2 Coverings of higher genus surfaces

We continue our study of ramified coverings started in Section 2.4.2 by considering ramified coverings whose base space Σ is an orientable compact surface of genus $g \geq 1$ rather than the sphere $\mathbb{CP}^1$.

The surface Σ now has a non-trivial homology group with $2g$ generators. We make a choice of a symplectic basis $(\mathcal{A}_h, \mathcal{B}_h)_{1 \leq h \leq g}$. We choose representatives (a_h, b_h) of these classes intersecting at a point P_0 only. This allows us to define, as in Section 2.7.2 a fundamental polygon $\mathring{\Sigma}$ for the surface. This will be more convenient to depict maps on high genus surfaces using these fundamental polygons. We can assume that the edges of the fundamental polygon correspond, in clockwise order, to

$$a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}.$$

Similarly, the fundamental group of Σ is also generated by the classes of $(a_h, b_h)_{1 \leq h \leq g}$. We abuse convention and identify them with a particular choice of representative for each of them.

We now define a map $\mathfrak{m}$ on Σ. Fix a point P in the interior of $\mathring{\Sigma}$. Let a_i' (resp. b_i') be a path going from P to the middle of a_i, and then from the middle of a_i^{-1} to P (resp. to the middle of b_i, and then from the middle of b_i^{-1}), and a_i. We assume that we chose them so that these path are non-intersecting. This defines a map $\mathfrak{m}$ whose underlying surface is homeomorphic to Σ. Said otherwise, if we denote by $\tilde{\Sigma}$ the universal cover of Σ, the paths (a_h, b_h) define a map $\tilde{\mathfrak{m}}$ in $\tilde{\Sigma}$. The map $\mathfrak{m}$ is the projection of the dual map to $\tilde{\mathfrak{m}}$.

Let $\pi\colon \hat{\Sigma} \to \Sigma$ be a ramified covering with set of branch points $R = \{P\}$. The map $\mathfrak{m}$ on Σ may be lifted to a map $\hat{\mathfrak{m}}$ on $\hat{\Sigma}$, as depicted in Figure 8.1.

In terms of monodromy representation, $\hat{\mathfrak{m}}$ is determined by one permutation for each of the $2g$ paths (a_i, b_i), and by one permutation for each branch point of π. Let d be the degree of π. We now explain how to obtain the permutational model for the map $\hat{\mathfrak{m}}$. Assume first that P is the only branch

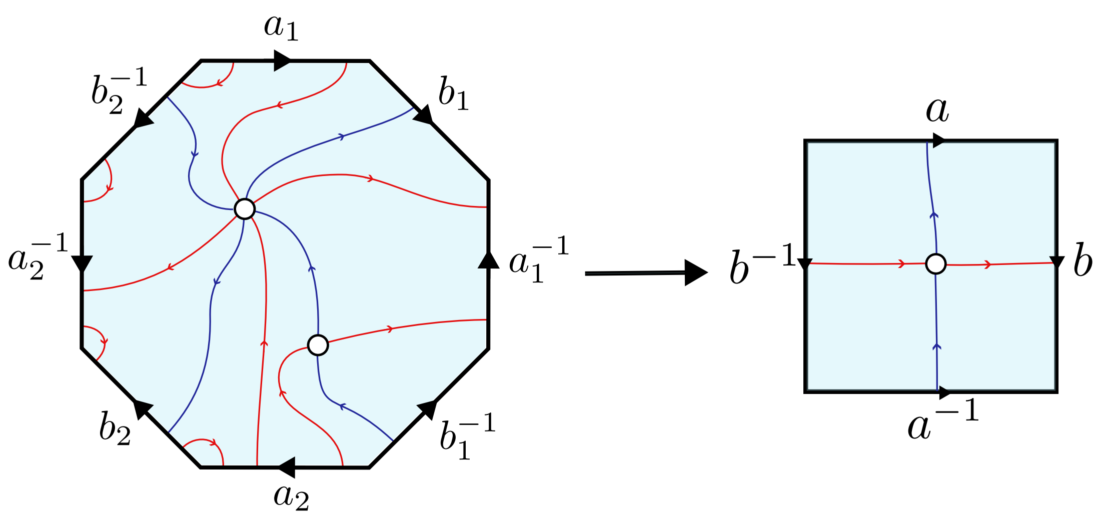


Figure 8.1: Covering of a torus by a double torus. The paths a_1' and b_1' and their lifts are depicted in blue and red.

point, with a small loop in the clockwise direction around P represented by π. Let ρ_i represent a_i, and σ_i represent b_i for $i \in [g]$. Note that we have

$$\pi = \sigma_g^{-1}\rho_g^{-1}\sigma_g\rho_g \cdots \sigma_1^{-1}\rho_1^{-1}\sigma_1\rho_1.$$

In terms of $\mathfrak{m}$, P is a vertex of degree $4g$. Since the map is oriented, the half-edges are oriented. Denote by

$$h_1, h_{g+1}, h_{2g+1}, h_{3g+1}, \ldots, h_g, h_{2g}, h_{3g}, h_{4g}$$

the half-edges around P in the clockwise direction, with h_1 part of a_1. Notice that with this choice, h_k and h_{2g+k} are the two half-edges of a_k' if $1 \leq k \leq g$ and of b_{k-g}' if $g+1 \leq k \leq 2g$. Using this labelling, in $\mathfrak{m}$ we have

$$\sigma_{\mathfrak{m}} = (1\; g+1\; 2g+1\; 3g+1\; 2\; g+2\; 2g+2\; 3g+2\; \ldots\ldots\; g\; 2g\; 3g\; 4g), \quad \text{and} \quad \alpha_{\mathfrak{m}} = \prod_{i=1}^{2g}(i\; 2g+i).$$

Let $\epsilon_{\mathfrak{m}} = (\epsilon(i))_{i\in[4m]}$ encode the orientation of the h_i's: $\epsilon(i) = +1$ if $1 \leq i \leq 2g$, and $\epsilon(i) = -1$ if $2g+1 \leq i \leq 4g$. Finally, let $t_{\mathfrak{m}} \colon [4g] \to [2g]$ encode the types, or color, of the half-edges. For $i \in [4g]$, we set

$$t_{\mathfrak{m}}(i) = i \mod 2g.$$

Note that $t_{\mathfrak{m}} \circ \alpha_{\mathfrak{m}} = t_{\mathfrak{m}}$, and $\epsilon(\alpha_{\mathfrak{m}}(i)) = -\epsilon(i)$ for all $i \in [4g]$. This give $\mathfrak{m}$ the structure of a oriented colored map.

For $i \in [4g]$, each half-edge h_i has d lifts. Assuming that we have labelled the sheets of the covering, we denote them by $\hat{h}_{i,j}, j \in [d]$. Let $k \in [g]$. An oriented edge from h_i to h_{i+2g} corresponding to a_i' lifts to d paths, between the half-edges $\hat{h}_{i,j}$ and $\hat{h}_{i+2g,\sigma_k^{-1}(j)}$ for all $j \in [d]$. Similarly, an oriented edge from h_i to h_{i+2g} corresponding to b_i' lifts to d paths, between the half-edges $\hat{h}_{i,j}$ and $\hat{h}_{i+2g,\rho_k(j)}$ for all $j \in [d]$. This remark allows us to give the permutational model of $\hat{\mathfrak{m}}$.

The map $\hat{h}$ is determined by the vertex permutation $\sigma_{\hat{\mathfrak{m}}}$ acting on the set of half-edges by

$$\sigma_{\hat{\mathfrak{m}}}(\hat{h}_{i,j}) = \begin{cases} \hat{h}_{\sigma_{\mathfrak{m}}(i),j} & \text{if } i \neq 4g \\ \hat{h}_{\sigma_{\mathfrak{m}}(i),\pi(j)} & \text{if } i = 4g. \end{cases}$$

and by the edge permutation $\alpha_{\hat{\mathfrak{m}}}$ acting on the set of half-edges by

$$\alpha_{\hat{\mathfrak{m}}}(\hat{h}_{i,j}) = \begin{cases} \hat{h}_{\alpha_{\mathfrak{m}}(i),\rho_k^{\epsilon(i)}(j)} & \text{if } 1 \leq k = t_{\mathfrak{m}}(i) \leq g \\ \hat{h}_{\alpha_{\mathfrak{m}}(i),\sigma_{k-g}^{-\epsilon(i)}(j)} & \text{if } g+1 \leq k = t_{\mathfrak{m}}(i) \leq 2g \end{cases}, \quad \text{for } i \in [4g], j \in [d].$$

The edge permutation $\alpha_{\hat{\mathfrak{m}}}$ is an involution without fixed point: assume that $t(i) \in [g]$,

$$\alpha_{\hat{\mathfrak{m}}}^2(\hat{h}_{i,j}) = \alpha_{\hat{\mathfrak{m}}}(\hat{h}_{\alpha_{\mathfrak{m}}(i),\rho_k^{\epsilon(i)}(j)}) = \hat{h}_{\alpha_{\mathfrak{m}}^2(i),\rho_k^{\epsilon\circ\alpha_{\mathfrak{m}}(i)}\circ\rho_k^{\epsilon(i)}(j)} = \hat{h}_{i,j}.$$

The fact that $\alpha_{\hat{\mathfrak{m}}}$ does not have fixed point follows from the fact that $\alpha_{\mathfrak{m}}$ does not either. The argument is similar if $t(i) > g$. We then define the orientation and coloring data $\boldsymbol{\epsilon}_{\hat{\mathfrak{m}}} = (\hat{\epsilon}(i,j))$ and $t_{\hat{\mathfrak{m}}} \colon [4g] \times [d] \to [2g]$: for all $i \in [4g], j \in [d]$,

$$\hat{\epsilon}(i,j) = \epsilon_i,$$
$$t_{\hat{\mathfrak{m}}}(i,j) = t_{\mathfrak{m}}(i).$$

The quadruplet $(\sigma_{\hat{\mathfrak{m}}}, \alpha_{\hat{\mathfrak{m}}}, \boldsymbol{\epsilon}_{\hat{\mathfrak{m}}}, t_{\hat{\mathfrak{m}}})$ determines the half-edge labelled, oriented, colored map $\hat{\mathfrak{m}}$.

The construction we just described can be seen as a generalization of the map-as-a-lift construction of Section 2.4.2. We now give a generalize the previous construction. It gives a higher genus generalization of the description of Hurwitz numbers as maps.

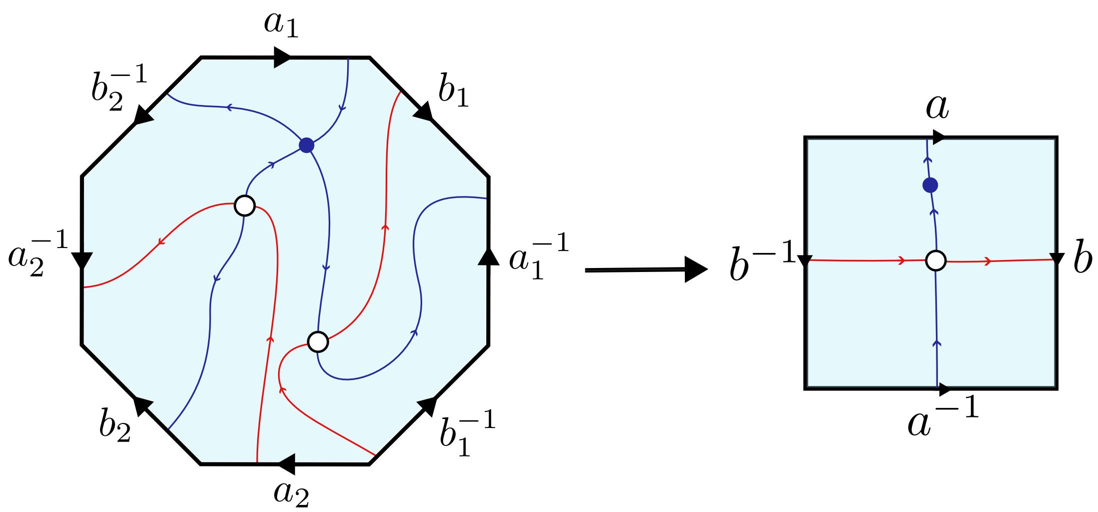


Figure 8.2: Lifting of the map $\mathfrak{m}$ on the torus to a map $\hat{\mathfrak{m}}$ on the double torus, in the case $r_1 = 1, r_2 = 0$.

Consider the same surface Σ, curves $(a_h, b_h)_{1 \le h \le g}$, and point P as before. Let $r_1, \ldots, r_{2g} \ge 0$ be integers. For all $i \in [g]$, we add:

- along a_i', between P and a_i, r_i vertices $v_i^{(1)}, \ldots, v_i^{(r_i)}$;
- along b_i', between P and b_i, r_{g+i} vertices $v_{i+g}^{(1)}, \ldots, v_{i+g}^{(r_i)}$.

This gives a map $\mathfrak{m}$ on Σ.

Assume that $\pi \colon \hat{\Sigma} \to \Sigma$ is ramified at P with any profile, and at the $v_i^{(j)}$'s with a simple profile. Let $\hat{\pi}, \pi_i^{(j)}$ be permutation representing P and the $v_i^{(j)}$'s. Assume that we chose the loops so that

$$\prod_{h=1}^{g} \sigma_h^{-1}\rho_h^{-1}\sigma_h\rho_h = \hat{\pi}\pi_{2g}^{(r_{2g})}\cdots\pi_{2g}^{(1)}\pi_{2g-1}^{(r_{2g-1})}\cdots\pi_{2g-1}^{(1)}\cdots\pi_1^{(r_1)}\cdots\pi_1^{(1)}. \tag{8.4}$$

Counting tuples of permutations (with a prescribed cyclic type) satisfying (8.4) is equivalent to computing Hurwitz numbers. We now explain how we might see such tuples as defining multicolored maps of unitary type.

In $\hat{\mathfrak{m}}$, we distinguish the white vertices (the preimages of P) and the colored vertices (the preimages of the $v_i^{(j)}$'s), with $v_i^{(j)}$ of color i, for all $j \in [r_i]$. With this coloring, the map $\hat{\mathfrak{m}}$ is a multicolored map of unitary type: the edges are consistently colored, the labels of the half-edges connected to white vertices are labelled by elements of

$$I = \{(i,j) \in [4g] \times [d]\}\,,$$

each preimage of a degree 4 colored vertex $v_i^{(j)}$ is numbered by j, and the edges between two such vertices have an orientation which mirror the monotonicity of the labels.

In fact, in some cases, the multicolored maps of unitary types can be seen as coverings over surfaces of genus $g \geq 1$. Such a fact has been shown by Novak [Nov25] in the case of the integral

$$\int_{\mathbb{U}(N)^2} \exp(zN \operatorname{Tr} UVU^*V^*)\mathrm{d}U\mathrm{d}V.$$

He can interpret the coefficient of the expansion of the logarithm of this integral in z and N in terms of generalized monotone Hurwitz numbers, counting covering whose base space is a torus with a monotonicity condition. An example of a covering counted by these numbers is displayed in Figure 8.2.

We now give for the sake of completeness the permutational model of such maps. Denote similarly as before by

$$h_1^{(0)}, h_{g+1}^{(0)}, h_{2g+1}^{(0)}, h_{3g+1}^{(0)}, \dots, h_g^{(0)}, h_{2g}^{(0)}, h_{3g}^{(0)}, h_{4g}^{(0)}$$

the half-edges around P in the clockwise direction, with $h_1^{(0)}$ part of a_1. Notice that with this choice, $h_k^{(0)}$ and h_{2g+k} are the first and last half-edges of a'_k if $1 \leq k \leq g$ and of b'_{k-g} if $g+1 \leq k \leq 2g$. Denote the other half-edges in a_k by $h_k^{(1)}, \dots, h_k^{(2r_k)}, h_k^{(2r_k+1)} = h_{2g+k}^{(0)}$.

When lifting the map $\mathfrak{m}$ to a map $\hat{\mathfrak{m}}$, each half-edge $h_i^{(p)}$ is lifted to d half-edges, which we denote by $\hat{h}_{i,j}^{(p)}$ for $j \in [d]$. The vertex permutation is then given by

$$\sigma_{\hat{\mathfrak{m}}}(\hat{h}_{i,j}^{(p)}) = \begin{cases} \hat{h}_{i+1,j}^{(0)} & \text{if } p = 0 \text{ and } i \neq 4g \\ \hat{h}_{1,\hat{\pi}(j)}^{(0)} & \text{if } p = 0 \text{ and } i = 4g \\ \hat{h}_{i,\pi_i^{(\frac{p+1}{2})}(j)}^{(p-1)} & \text{if } p \neq 0 \text{ and } p \text{ odd} \\ \hat{h}_{i,j}^{(p+1)} & \text{if } p \neq 0 \text{ and } p \text{ even}\,. \end{cases} \qquad \text{for all } i \in [4g], j \in [d], p \in \{0, \dots, r_{(i \mod 2g)}\}.$$

Consider an edge e, part of a'_i if $1 \leq i \leq g$ or part of b'_{i-g} if $g+1 \leq i \leq 2g$. The edge e is made of the two half-edges $h_i^{(2p)}$ and $h_i^{(2p+1)}$ with $p \in \{0, \dots, r_i - 1\}$. This edge is lifted to d edges, which are made of $\hat{h}_{i,j}^{(p)}$ and $\hat{h}_{i,j}^{(p+1)}$ for $j \in [d]$. The edge made of $h_i^{(2r_i)}$ and $h_i^{(2r_i+1)} = h_{i+2g}^{(0)}$ is lifted to d edges, made of

- $\hat{h}_{i,j}^{(2r_i)}$ and $\hat{h}_{i,\sigma_i^{-1}(j)}^{(2r_i+1)} = \hat{h}_{i+2g,\sigma_i^{-1}(j)}^{(0)}$, for $j \in [d]$, if $1 \leq i \leq g$;
- $\hat{h}_{i,j}^{(2r_i)}$ and $\hat{h}_{i,\rho_{i-g}(j)}^{(2r_i+1)} = \hat{h}_{i+2g,\rho_{i-g}(j)}^{(0)}$, for $j \in [d]$, if $g+1 \leq i \leq 2g$.

Using this remark, we may write the edge permutation $\alpha_{\hat{\mathfrak{m}}}$ as follows: for each $i \in [2g]$, $j \in [d]$,

$$\alpha_{\hat{\mathfrak{m}}}(\hat{h}_{i,j}^{(p)}) = \begin{cases} \hat{h}_{i,j}^{(p+1)} & \text{if } p \text{ is even and } p \leq 2r_i - 1 \\ \hat{h}_{i,j}^{(p-1)} & \text{if } p \text{ is odd and } p \leq 2r_i - 1 \\ \hat{h}_{i+2g,\sigma_i^{-1}(j)}^{(0)} & \text{if } p = 2r_i \text{ and } 1 \leq i \leq g \\ \hat{h}_{i+2g,\rho_i(j)}^{(0)} & \text{if } p = 2r_i \text{ and } g+1 \leq i \leq 2g. \end{cases}$$

We then complete the definition of $\alpha_{\hat{\mathfrak{m}}}$ so that it is an involution.

A permutational model for a map of unitary type can be deduced: we set

$$\pi_{\hat{\mathfrak{m}}}(\hat{h}^{(0)}_{i,j}) = \begin{cases} \hat{h}^{(0)}_{i+2g,\sigma_i^{-1}\pi_i^{(r_i)}\cdots\pi_i^{(1)}(j)} & \text{if } 1 \leq i \leq g \\ \hat{h}^{(0)}_{i+2g,\rho_i\pi_i^{(r_i)}\cdots\pi_i^{(1)}(j)} & \text{if } g+1 \leq i \leq 2g \\ \hat{h}^{(0)}_{i-2g,\sigma_i^{-1}(j)} & \text{if } 2g+1 \leq i \leq 3g \\ \hat{h}^{(0)}_{i-2g,\rho_i(j)} & \text{if } 3g+1 \leq i \leq 4g \end{cases},$$

and for all $i \in [2g]$, $p \in [r_i]$:

$$\tau_i^{(p)} = (\hat{h}^{(0)}_{i,j}\,\hat{h}^{(0)}_{i,\pi_i^{(p)}(j)}).$$

The permutations $\pi_{\hat{\mathfrak{m}}}$ and $(\tau_i^{(p)})_{i\in[2g],p\in[r_i]}$ act on the set of half-edges

$$\hat{I} = \left\{\hat{h}^{(0)}_{i,j} : (i,j) \in I\right\},$$

that can be identified with I. We can check that the permutations $\sigma_{\hat{\mathfrak{m}}}|_{\hat{I}}, \pi_{\hat{\mathfrak{m}}}, (\tau_i^{(p)})$ are the permutational model for the map of unitary type $\hat{\mathfrak{m}}$.

Bibliography

[AAM14] A. Abdesselam, G. W. Anderson, and A. Miller. *Tridiagonalized GUE Matrices Are a Matrix Model for Labeled Mobiles.* 2014. URL: http://arxiv.org/abs/1404.7415 (visited on 06/20/2023). Pre-published.

[ADK21a] J. Alt, R. Ducatez, and A. Knowles. "Delocalization Transition for Critical Erdős–Rényi Graphs". In: *Communications in Mathematical Physics* 388.1 (2021), pp. 507–579.

[ADK21b] J. Alt, R. Ducatez, and A. Knowles. "Extremal Eigenvalues of Critical Erdős–Rényi Graphs". In: *The Annals of Probability* 49.3 (2021), pp. 1347–1401.

[ADK23] J. Alt, R. Ducatez, and A. Knowles. "Poisson Statistics and Localization at the Spectral Edge of Sparse Erdős–Rényi Graphs". In: *The Annals of Probability* 51.1 (2023), pp. 277–358.

[ADK24] J. Alt, R. Ducatez, and A. Knowles. "Localized Phase for the Erdős–Rényi Graph". In: *Communications in Mathematical Physics* 405.1 (2024), p. 9.

[AGZ10] G. W. Anderson, A. Guionnet, and O. Zeitouni. *An Introduction to Random Matrices.* Cambridge University Press, 2010. 507 pp.

[AHvM99] M. Adler, E. Horozov, and P. van Moerbeke. "The Pfaff Lattice and Skew-Orthogonal Polynomials". In: *International Mathematics Research Notices* 1999.11 (1999), pp. 569–588.

[Ale+18] A. Alexandrov et al. "Weighted Hurwitz Numbers and Topological Recursion: An Overview". In: *Journal of Mathematical Physics* 59.8 (2018), p. 081102.

[AM93] M. Aizenman and S. Molchanov. "Localization at Large Disorder and at Extreme Energies: An Elementary Derivations". In: *Communications in Mathematical Physics* 157.2 (1993), pp. 245–278.

[And58] P. W. Anderson. "Absence of Diffusion in Certain Random Lattices". In: *Physical Review* 109.5 (1958), pp. 1492–1505.

[APS01] S. Albeverio, L. Pastur, and M. Shcherbina. "On the 1/n Expansion for Some Unitary Invariant Ensembles of Random Matrices". In: *Communications in Mathematical Physics* 224.1 (2001), pp. 271–305.

[Arq86] D. Arques. "Les hypercartes planaires sont des arbres tres bien etiquetes". In: *Discrete Mathematics* 58.1 (1986), pp. 11–24.

[ASM02] M. Adler, T. Shiota, and P. van Moerbeke. "Pfaff τ-functions". In: *Mathematische Annalen* 322 (2002).

[AW15] M. Aizenman and S. Warzel. *Random Operators: Disorder Effects on Quantum Spectra and Dynamics.* Graduate Studies in Mathematics volume 168. Providence, Rhode Island: American Mathematical Society, 2015. 326 pp.

[AW95] J. Ambjørn and Y. Watabiki. "Scaling in Quantum Gravity". In: *Nuclear Physics B* 445.1 (1995), pp. 129–142.

[BB24] G. Borot and T. Buc-d'Alché. "Fay Identities of Pfaffian Type for Hyperelliptic Curves". In: *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications* 20 (2024), p. 054.

[BBT10] O. Babelon, D. Bernard, and M. Talon. *Introduction to Classical Integrable Systems.* Cambridge Monographs on Mathematical Physics. Cambridge: Cambridge Univ. Press, 2010. 602 pp.

[BC94] E. A. Bender and E. R. Canfield. “The Number of Degree-Restricted Rooted Maps on the Sphere”. In: *SIAM Journal on Discrete Mathematics* 7.1 (1994), pp. 9–15.

[BD99] E. Brézin and N. Deo. “Correlations and Symmetry Breaking in Gapped Matrix Models”. In: *Physical Review E* 59.4 (1999), pp. 3901–3910.

[BDE00] G. Bonnet, F. David, and B. Eynard. “Breakdown of Universality in Multi-Cut Matrix Models”. In: *Journal of Physics A: Mathematical and General* 33.38 (2000), p. 6739.

[BDG02] J. Bouttier, P. Di Francesco, and E. Guitter. “Census of Planar Maps: From the One-Matrix Model Solution to a Combinatorial Proof”. In: *Nuclear Physics B* 645.3 (2002), pp. 477–499.

[BDG03a] J. Bouttier, P. Di Francesco, and E. Guitter. “Geodesic Distance in Planar Graphs”. In: *Nuclear Physics B* 663.3 (2003), pp. 535–567.

[BDG03b] J. Bouttier, P. Di Francesco, and E. Guitter. “Statistics of Planar Graphs Viewed from a Vertex: A Study via Labeled Trees”. In: *Nuclear Physics B* 675.3 (2003), pp. 631–660.

[BDG04] J. Bouttier, P. Di Francesco, and E. Guitter. “Planar Maps as Labeled Mobiles”. In: *The Electronic Journal of Combinatorics* (2004), R69–R69.

[BE12] G. Borot and B. Eynard. “Geometry of Spectral Curves and All Order Dispersive Integrable System”. In: *Symmetry, Integrability and Geometry: Methods and Applications* (2012).

[BEO15] G. Borot, B. Eynard, and N. Orantin. “Abstract Loop Equations, Topological Recursion and New Applications”. In: *Communications in Number Theory and Physics* 9.1 (2015), pp. 51–187.

[Ber03] M. Bertola. “Free Energy of the Two-Matrix Model/dToda Tau-Function”. In: *Nuclear Physics B* 669.3 (2003), pp. 435–461.

[Ber06] M. Bertola. *Riemann Surfaces and Theta Functions.* 2006. Pre-published.

[Ber11] M. Bertola. “Boutroux Curves with External Field: Equilibrium Measures without a Variational Problem”. In: *Analysis and Mathematical Physics* 1.2 (2011), pp. 167–211.

[Bet10] J. Bettinelli. “Scaling Limits for Random Quadrangulations of Positive Genus”. In: *Electronic Journal of Probability* 15 (2010), pp. 1594–1644.

[Bet22] J. Bettinelli. “A Bijection for Nonorientable General Maps”. In: *Annales de l’Institut Henri Poincaré D* 9.4 (2022), pp. 733–791.

[BFG14] J. Bouttier, É. Fusy, and E. Guitter. “On the Two-Point Function of General Planar Maps and Hypermaps”. In: *Annales de l’Institut Henri Poincaré D, Combinatorics, Physics and their Interactions* 1.3 (2014), pp. 265–306.

[BG12] J. Bouttier and E. Guitter. “Planar Maps and Continued Fractions”. In: *Communications in Mathematical Physics* 309.3 (2012), pp. 623–662.

[BG13] G. Borot and A. Guionnet. “Asymptotic Expansion of β Matrix Models in the One-cut Regime”. In: *Communications in Mathematical Physics* 317.2 (2013), pp. 447–483.

[BG14] J. Bouttier and E. Guitter. “On Irreducible Maps and Slices”. In: *Combinatorics, Probability and Computing* 23.6 (2014), pp. 914–972.

[BG18] G. Borot and E. Garcia-Failde. “Simple Maps, Hurwitz Numbers, and Topological Recursion”. 2018.

[BG24] G. Borot and A. Guionnet. “Asymptotic Expansion of Matrix Models in the Multi-Cut Regime”. In: *Forum of Mathematics, Sigma* 12 (2024), e13.

[BGK14] G. Borot, A. Guionnet, and K. K. Kozlowski. “Large-N Asymptotic Expansion for Mean Field Models with Coulomb Gas Interaction”. 2014.

[BH16] E. Brézin and S. Hikami. *Random Matrix Theory with an External Source.* Vol. 19. SpringerBriefs in Mathematical Physics. Singapore: Springer Singapore, 2016.

[BIZ80] D. Bessis, C. Itzykson, and J.-B. Zuber. “Quantum Field Theory Techniques in Graphical Enumeration”. In: *Advances in Applied Mathematics* 1.2 (1980), pp. 109–157.

[BK24] T. Buc-d’Alché and A. Knowles. *Semi-Localization in the Generalized Random Graph Model.* 2024. URL: https://hal.science/hal-04806113. Pre-published.

[BK87] D. V. Boulatov and V. A. Kazakov. “The Ising Model on a Random Planar Lattice: The Structure of the Phase Transition and the Exact Critical Exponents”. In: *Physics Letters B* 186.3–4 (1987), pp. 379–384.

[BLM13] S. Boucheron, G. Lugosi, and P. Massart. *Concentration Inequalities: A Nonasymptotic Theory of Independence.* Oxford: Oxford University Press, 2013. 496 pp.

[BM22] J. Bettinelli and G. Miermont. *Compact Brownian Surfaces II. Orientable Surfaces.* 2022. URL: http://arxiv.org/abs/2212.12511 (visited on 03/28/2025). Pre-published.

[Bon15] M. Bona, ed. *Handbook of Enumerative Combinatorics.* New York: Chapman and Hall/CRC, 2015. 1086 pp.

[Bor+11] G. Borot et al. “A Matrix Model for Simple Hurwitz Numbers, and Topological Recursion”. In: *Journal of Geometry and Physics* 61.2 (2011), pp. 522–540.

[Bor+21] G. Borot et al. “Analytic Theory of Higher Order Free Cumulants”. In: *arXiv.org* (2021).

[Bré+78] E. Brézin et al. “Planar Diagrams”. In: *Communications in Mathematical Physics* 59.1 (1978), pp. 35–51.

[BS03] M. Bousquet-Melou and G. Schaeffer. *The Degree Distribution in Bipartite Planar Maps: Applications to the Ising Model.* 2003. URL: http://arxiv.org/abs/math/0211070 (visited on 03/25/2025). Pre-published.

[BS06] A. Borodin and E. Strahov. “Averages of Characteristic Polynomials in Random Matrix Theory”. In: *Communications on Pure and Applied Mathematics* 59.2 (2006), pp. 161–253.

[Buc24] T. Buc-d’Alché. “Topological Expansion of Unitary Integrals and Maps”. In: *Annales de l’Institut Henri Poincaré D* (2024).

[CD17] G. Chapuy and M. Dołęga. “A Bijection for Rooted Maps on General Surfaces”. In: *Journal of Combinatorial Theory, Series A* 145 (2017), pp. 252–307.

[CD22] G. Chapuy and M. Dołęga. “Non-orientable branched coverings, b-Hurwitz numbers, and positivity for multiparametric Jack expansions”. In: *Advances in Mathematics* 409 (2022), p. 108645.

[CDG24] N. A. Cook, R. Ducatez, and A. Guionnet. *Full Large Deviation Principles for the Largest Eigenvalue of Sub-Gaussian Wigner Matrices.* 2024. URL: http://arxiv.org/abs/2302.14823 (visited on 03/28/2025). Pre-published.

[CE06] L. Chekhov and B. Eynard. “Matrix Eigenvalue Model: Feynman Graph Technique for All Genera”. In: *Journal of High Energy Physics* 2006.12 (2006), pp. 026–026.

[CGL23] B. Collins, R. Gurau, and L. Lionni. “The Tensor Harish-Chandra–Itzykson–Zuber Integral I: Weingarten Calculus and a Generalization of Monotone Hurwitz Numbers”. In: *Journal of the European Mathematical Society* (2023).

[CGM09] B. Collins, A. Guionnet, and E. Maurel-Segala. “Asymptotics of Unitary and Othogonal Matrix Integrals”. In: *Advances in Mathematics* 222.1 (2009), pp. 172–215.

[CGM15] T. Claeys, T. Grava, and K. D. T.-R. McLaughlin. “Asymptotics for the Partition Function in Two-Cut Random Matrix Models”. In: *Communications in Mathematical Physics* 339.2 (2015), pp. 513–587.

[Cha+23] C. Charlier et al. *Asymptotics of Hankel Determinants with a Multi-Cut Regular Potential and Fisher-Hartwig Singularities*. 2023. URL: http://arxiv.org/abs/2111.08395 (visited on 02/26/2025). Pre-published.

[Cha09] G. Chapuy. “Asymptotic Enumeration of Constellations and Related Families of Maps on Orientable Surfaces”. In: *Combinatorics, Probability and Computing* 18.4 (2009), pp. 477–516.

[CHS18] B. Collins, T. Hasebe, and N. Sakuma. “Free Probability for Purely Discrete Eigenvalues of Random Matrices”. In: *Journal of the Mathematical Society of Japan* 70.3 (2018).

[Cic82] G. M. Cicuta. “Topological expansion for SO_N and Sp_{2N} gauge theories”. In: *Lettere al Nuovo Cimento (1971-1985)* 35.3 (1982), pp. 87–92.

[CMN22] B. Collins, S. Matsumoto, and J. Novak. “The Weingarten Calculus”. In: *Notices of the American Mathematical Society* 69.05 (2022), p. 1.

[CMP17] J. Carlson, S. Müller-Stach, and C. Peters. *Period Mappings and Period Domains*. 2nd ed. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2017.

[CMS09] G. Chapuy, M. Marcus, and G. Schaeffer. “A Bijection for Rooted Maps on Orientable Surfaces”. In: *SIAM Journal on Discrete Mathematics* 23.3 (2009), pp. 1587–1611.

[Col+07] B. Collins et al. “Second Order Freeness and Fluctuations of Random Matrices. III: Higher Order Freeness and Free Cumulants.” In: *Documenta Mathematica* 12 (2007), pp. 1–70.

[Col03] B. Collins. “Moments and Cumulants of Polynomial Random Variables on Unitary Groups, the Itzykson-Zuber Integral, and Free Probability”. In: *International Mathematics Research Notices* 2003.17 (2003), pp. 953–982.

[Cor75] R. Cori. “Un code pour les graphes planaires et ses applications”. In: *Société mathématique de France*. Astérisque 27 (1975).

[CPZ24] L. Cassia, V. Posch, and M. Zabzine. *β-ensembles and higher genera Catalan numbers*. 2024. URL: http://arxiv.org/abs/2310.05626 (visited on 02/07/2024). Pre-published.

[CS02] P. Chassaing and G. Schaeffer. “Random Planar Lattices and Integrated SuperBrownian Excursion”. In: Colloquium on Mathematics and Computer Science: Algorithms, Trees, Combinatorics and Probabilities. Birkhauser, 2002, p. 123.

[CŚ06] B. Collins and P. Śniady. “Integration with Respect to the Haar Measure on Unitary, Orthogonal and Symplectic Group”. In: *Communications in Mathematical Physics* 264.3 (2006), pp. 773–795.

[CV81] R. Cori and B. Vauquelin. “Planar Maps Are Well Labeled Trees”. In: *Canadian Journal of Mathematics* 33.5 (1981), pp. 1023–1042.

[DDM14] N. Do, A. Dyer, and D. V. Mathews. *Topological Recursion and a Quantum Curve for Monotone Hurwitz Numbers*. 2014. URL: http://arxiv.org/abs/1408.3992 (visited on 09/17/2023). Pre-published.

[DE02] I. Dumitriu and A. Edelman. “Matrix Models for Beta Ensembles”. In: *Journal of Mathematical Physics* 43.11 (2002), pp. 5830–5847.

[Dei+97] P. Deift et al. “Asymptotics for Polynomials Orthogonal with Respect to Varying Exponential Weights”. In: *International Mathematics Research Notices* 1997.16 (1997), pp. 759–782.

[Dei+99a] P. Deift et al. “Strong Asymptotics of Orthogonal Polynomials with Respect to Exponential Weights”. In: *Communications on Pure and Applied Mathematics* 52.12 (1999), pp. 1491–1552.

[Dei+99b] P. Deift et al. “Uniform Asymptotics for Polynomials Orthogonal with Respect to Varying Exponential Weights and Applications to Universality Questions in Random Matrix Theory”. In: *Communications on Pure and Applied Mathematics* 52.11 (1999), pp. 1335–1425.

[Dei00] P. Deift. *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*. Courant Lecture Notes in Mathematics 3. Providence, R.I: American Mathematical Society, 2000. 261 pp.

[DF16] P. Diaconis and P. J. Forrester. “A. Hurwitz and the Origins of Random Matrix Theory in Mathematics”. 2016.

[DGZ95] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin. “2D Gravity and Random Matrices”. In: *Physics Reports* 254.1–2 (1995), pp. 1–133.

[DHL25] M. Duits, N. Hayford, and S.-Y. Lee. *The Ising Model Coupled to 2D Gravity: Genus Zero Partition Function*. 2025. URL: http://arxiv.org/abs/2405.03259 (visited on 03/25/2025). Pre-published.

[Dia87] P. Diaconis. “Application of the Method of Moments in Probability and Statistics”. In: *Proceedings of Symposia in Applied Mathematics*. Ed. by H. Landau. Vol. 37. Providence, Rhode Island: American Mathematical Society, 1987, pp. 125–142.

[DiF05] P. Di Francesco. “Geodesic Distance in Planar Graphs: An Integrable Approach”. In: *The Ramanujan Journal* 10.2 (2005), pp. 153–186.

[DP21] G. Dubach and Y. Peled. “On Words of Non-Hermitian Random Matrices”. In: *The Annals of Probability* 49.4 (2021), pp. 1886–1916.

[DV02] R. Dijkgraaf and C. Vafa. “Matrix Models, Topological Strings, and Supersymmetric Gauge Theories”. In: *Nuclear Physics B* 644.1 (2002), pp. 3–20.

[Dys62a] F. J. Dyson. “Statistical Theory of the Energy Levels of Complex Systems. I”. In: *Journal of Mathematical Physics* 3.1 (1962), pp. 140–156.

[Dys62b] F. J. Dyson. “The Threefold Way. Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics”. In: *Journal of Mathematical Physics* 3.6 (1962), pp. 1199–1215.

[Edm60] J. R. Edmonds. “A Combinatorial Representation for Polyhedral Surfaces”. In: *Notices of the American Mathematical Society* 7.48 (1960), p. 646.

[EKR18] B. Eynard, T. Kimura, and S. Ribault. *Random Matrices*. Version 2. 2018. URL: http://arxiv.org/abs/1510.04430 (visited on 07/21/2022). Pre-published.

[EM03] N. M. Ercolani and K. D. T.-R. McLaughlin. “Asymptotics of the Partition Function for Random Matrices via Riemann-Hilbert Techniques and Applications to Graphical Enumeration”. In: *IMRN. International Mathematics Research Notices* 2003.14 (2003), pp. 755–820.

[EM08] F. Evers and A. D. Mirlin. “Anderson Transitions”. In: *Reviews of Modern Physics* 80.4 (2008), pp. 1355–1417.

[EM98] B. Eynard and M. L. Mehta. “Matrices Coupled in a Chain. I. Eigenvalue Correlations”. In: *Journal of Physics A: Mathematical and General* 31.19 (1998), pp. 4449–4456.

[EO07a] B. Eynard and N. Orantin. “Invariants of Algebraic Curves and Topological Expansion”. In: *Communications in Number Theory and Physics* 1.2 (2007), pp. 347–452.

[EO07b] B. Eynard and N. Orantin. *Weil-Petersson Volume of Moduli Spaces, Mirzakhani’s Recursion and Matrix Models*. 2007. URL: http://arxiv.org/abs/0705.3600 (visited on 03/06/2025). Pre-published.

[EO08] B. Eynard and N. Orantin. "Algebraic Methods in Random Matrices and Enumerative Geometry". 2008.

[EO24] B. Eynard and S. Oukassi. *Hirota, Fay and Geometry.* 2024. URL: http://arxiv.org/abs/2401.08317 (visited on 01/18/2024). Pre-published.

[ER59] P. Erdős and A. Rényi. "On Random Graphs. I." In: *Publicationes Mathematicae Debrecen* 6.3–4 (1959), pp. 290–297.

[Eyn05] B. Eynard. "Topological Expansion for the 1-Hermitian Matrix Model Correlation Functions". In: *Journal of High Energy Physics* 2004.11 (2005), p. 031.

[Eyn09] B. Eynard. "Large N Expansion of Convergent Matrix Integrals, Holomorphic Anomalies, and Background Independence". In: *Journal of High Energy Physics* 2009.03 (2009), pp. 003–003.

[Eyn11] B. Eynard. "Formal Matrix Integrals and Combinatorics of Maps". In: *Random Matrices, Random Processes and Integrable Systems.* Ed. by J. Harnad. CRM Series in Mathematical Physics. New York, NY: Springer, 2011, pp. 415–442.

[Eyn14] B. Eynard. "Invariants of Spectral Curves and Intersection Theory of Moduli Spaces of Complex Curves". In: *Communications in Number Theory and Physics* 8.3 (2014), pp. 541–588.

[Fay73] J. D. Fay. *Theta Functions on Riemann Surfaces.* Vol. 352. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 1973.

[FIK91] A. S. Fokas, A. R. Its, and A. V. Kitaev. "Discrete Painlevé Equations and Their Appearance in Quantum Gravity". In: *Communications in Mathematical Physics* 142.2 (1991), pp. 313–344.

[FK92] H. M. Farkas and I. Kra. *Riemann Surfaces.* Vol. 71. Graduate Texts in Mathematics. New York, NY: Springer, 1992.

[FM91] Y. V. Fyodorov and A. D. Mirlin. "Scaling properties of localization in random band matrices: A σ-model approach". In: *Physical Review Letters* 67.18 (1991), pp. 2405–2409.

[Fol95] G. B. Folland. *A Course in Abstract Harmonic Analysis.* Studies in Advanced Mathematics. Boca Raton, Fla.: CRC Press, 1995. 276 pp.

[FS13] P. Flajolet and R. Sedgewick. *Analytic Combinatorics.* 4th pr. Cambridge: Cambridge Univ. Press, 2013. 810 pp.

[FS83] J. Fröhlich and T. Spencer. "Absence of Diffusion in the Anderson Tight Binding Model for Large Disorder or Low Energy". In: *Communications in Mathematical Physics* 88.2 (1983), pp. 151–184.

[Ger+91] A. Gerasimov et al. "Matrix Models of Two-Dimensional Gravity and Toda Theory". In: *Nuclear Physics B* 357.2 (1991), pp. 565–618.

[GGN11] I. P. Goulden, M. Guay-Paquet, and J. Novak. "Monotone Hurwitz Numbers and the HCIZ Integral II". 2011.

[GGN14] I. P. Goulden, M. Guay-Paquet, and J. Novak. "Monotone Hurwitz Numbers and the HCIZ Integral". In: *Annales Mathématiques Blaise Pascal* 21.1 (2014), pp. 71–89.

[GHJ01] I. P. Goulden, J. L. Harer, and D. M. Jackson. "A Geometric Parametrization for the Virtual Euler Characteristics of the Moduli Spaces of Real and Complex Algebraic Curves". In: *Transactions of the American Mathematical Society* 353.11 (2001), pp. 4405–4427.

[GJ96] I. P. Goulden and D. M. Jackson. "Connection Coefficients, Matchings, Maps and Combinatorial Conjectures for Jack Symmetric Functions". In: *Transactions of the American Mathematical Society* 348.3 (1996), pp. 873–892.

[GJ97] I. P. Goulden and D. M. Jackson. “Maps in Locally Orientable Surfaces and Integrals Over Real Symmetric Surfaces”. In: *Canadian Journal of Mathematics* 49.5 (1997), pp. 865–882.

[GM06] A. Guionnet and E. Maurel-Segala. “Combinatorial Aspects of Matrix Models”. In: *ALEA. Latin American Journal of Probability and Mathematical Statistics* 1 (2006), pp. 241–279.

[GM07] A. Guionnet and E. Maurel-Segala. “Second Order Asymptotics for Matrix Models”. In: *The Annals of Probability* 35.6 (2007).

[GN15] Alice Guionnet and Jonathan Novak. “Asymptotics of Unitary Multimatrix Models: The Schwinger–Dyson Lattice and Topological Recursion”. In: *Journal of Functional Analysis* 268.10 (2015), pp. 2851–2905.

[GR01] C. Godsil and G. Royle. *Algebraic Graph Theory*. Graduate Texts in Mathematics 207. New York, NY: Springer, 2001. 443 pp.

[Gru12] S. Grushevsky. “The Schottky Problem”. In: *Current Developments in Algebraic Geometry*. Ed. by L. Caporaso et al. Mathematical Sciences Research Institute Publications 59. Cambridge: Cambridge Univ. Press, 2012, pp. 129–164.

[Gui04] A. Guionnet. “First Order Asymptotics of Matrix Integrals; A Rigorous Approach Towards the Understanding of Matrix Models”. In: *Communications in Mathematical Physics* 244.3 (2004), pp. 527–569.

[Hal00] A. Hald. “The Early History of the Cumulants and the Gram-Charlier Series”. In: *International Statistical Review / Revue Internationale de Statistique* 68.2 (2000), pp. 137–153.

[Har57] Harish-Chandra. “Differential Operators on a Semisimple Lie Algebra”. In: *American Journal of Mathematics* 79.1 (1957), pp. 87–120.

[HM23] E. Hiesmayr and T. McKenzie. *The Spectral Edge of Constant Degree Erdős-Rényi Graphs*. 2023. URL: http://arxiv.org/abs/2309.11007 (visited on 01/17/2024). Pre-published.

[HP14] F. Hiai and D. Petz. *Semicircle Law, Free Random Variables and Entropy*. Mathematical Surveys and Monographs v.77. Providence: American Mathematical Society, 2014. 389 pp.

[Hur91] A. Hurwitz. “Ueber Riemann’sche Flächen mit gegebenen Verzweigungspunkten”. In: *Mathematische Annalen* 39.1 (1891), pp. 1–60.

[Hur97] A. Hurwitz. “Über die Erzeugung der Invarianten durch Integration”. In: *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1897 (1897), pp. 71–2.

[Hus22] J. Husson. “Large Deviations for the Largest Eigenvalue of Matrices with Variance Profiles”. In: *Electronic Journal of Probability* 27 (2022), pp. 1–44.

[HZ86] J. Harer and D. Zagier. “The Euler Characteristic of the Moduli Space of Curves”. In: *Inventiones Mathematicae* 85.3 (1986), pp. 457–485.

[Iss18] L. Isserlis. “On a Formula for the Product-Moment Coefficient of Any Order of a Normal Frequency Distribution in Any Number of Variables”. In: *Biometrika* 12.1–2 (1918), pp. 134–139.

[IZ80] C. Itzykson and J.-B. Zuber. “The Planar Approximation. II”. In: *Journal of Mathematical Physics* 21.3 (1980), pp. 411–421.

[Jac70] H. Jack. “A Class of Symmetric Polynomials with a Parameter”. In: *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* 69.1 (1970), pp. 1–18.

[Jan07] S. Janson. “Brownian Excursion Area, Wright’s Constants in Graph Enumeration, and Other Brownian Areas”. In: *Probability Surveys* 4 (none 2007).

[Joh12] P. Johnson. “Hurwitz Numbers, Ribbon Graphs, and Tropicalization”. In: *Contemp. Math* 580 (2012), pp. 55–72.

[Joh98] K. Johansson. “On Fluctuations of Eigenvalues of Random Hermitian Matrices”. In: *Duke Mathematical Journal* 91.1 (1998).

[Jur91] J. Jurkiewicz. “Chaotic Behaviour in One-Matrix Models”. In: *Physics Letters B* 261.3 (1991), pp. 260–268.

[Kad97] K. W. J. Kadell. “The Selberg–Jack Symmetric Functions”. In: *Advances in Mathematics* 130.1 (1997), pp. 33–102.

[Kaz86] V. A. Kazakov. “Ising Model on a Dynamical Planar Random Lattice: Exact Solution”. In: *Physics Letters A* 119.3 (1986), pp. 140–144.

[KM16] M. Kornyik and G. Michaletzky. “Wigner Matrices, the Moments of Roots of Hermite Polynomials and the Semicircle Law”. In: *Journal of Approximation Theory* 211 (2016), pp. 29–41.

[Kre72] G. Kreweras. “Sur les partitions non croisees d’un cycle”. In: *Discrete Mathematics* 1.4 (1972), pp. 333–350.

[Kri77] I. M. Krichever. “Methods of Algebraic Geometry in the Theory of Nonlinear Equations”. In: *Russian Mathematical Surveys* 32.6 (1977), p. 185.

[KZ15] S. Kharchev and A. Zabrodin. “Theta Vocabulary I”. Version 2. In: *Journal of Geometry and Physics* 94 (2015), pp. 19–31.

[KZ22] V. Kazakov and Z. Zheng. “Analytic and Numerical Bootstrap for One-Matrix Model and "Unsolvable" Two-Matrix Model”. In: *Journal of High Energy Physics* 2022.6 (2022), p. 30.

[LaC09] M. A. LaCroix. “The Combinatorics of the Jack Parameter and the Genus Series for Topological Maps”. Ontario, Canada: University of Waterloo, 2009.

[Lan11] S. K. Lando. “Hurwitz Numbers: On the Edge Between Combinatorics and Geometry”. In: *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010).* Proceedings of the International Congress of Mathematicians 2010 (ICM 2010). Hyderabad, India: Published by Hindustan Book Agency (HBA), India. WSPC Distribute for All Markets Except in India, 2011, pp. 2444–2470.

[Lee12] J. M. Lee. *Introduction to Smooth Manifolds.* Graduate Texts in Mathematics. New York, NY: Springer, 2012.

[LeG13] J.-F. Le Gall. “Uniqueness and Universality of the Brownian Map”. In: *The Annals of Probability* 41.4 (2013), pp. 2880–2960.

[LS13] JO. Lee and K. Schnelli. “Local Deformed Semicircle Law and Complete Delocalization for Wigner Matrices with Random Potential”. In: *Journal of Mathematical Physics* 54.10 (2013), p. 103504.

[LS16] JO. Lee and K. Schnelli. “Extremal Eigenvalues and Eigenvectors of Deformed Wigner Matrices”. In: *Probability Theory and Related Fields* 164.1 (2016), pp. 165–241.

[LZ04] S. K. Lando and A. Zvonkin. *Graphs on Surfaces and Their Applications.* Ed. by R. V. Gamkrelidze and V. A. Vassiliev. Encyclopaedia of Mathematical Sciences, Low Dimensional Topology. Berlin Heidelberg: Springer-Verlag, 2004.

[Mac15] I. G. Macdonald. *Symmetric Functions and Hall Polynomials.* Second Edition, Second Edition. Oxford Classic Texts in the Physical Sciences. Oxford, New York: Oxford University Press, 2015. 488 pp.

[Mau06] E. Maurel-Segala. “High Order Expansion of Matrix Models and Enumeration of Maps”. 2006.

[Meh04] M. L. Mehta. *Random Matrices.* Elsevier, 2004. 707 pp.

[Meh81] M. L. Mehta. “A Method of Integration over Matrix Variables”. In: *Communications in Mathematical Physics* 79.3 (1981), pp. 327–340.

[Mie13] G. Miermont. "The Brownian Map Is the Scaling Limit of Uniform Random Plane Quadrangulations". In: *Acta Mathematica* 210.2 (2013), pp. 319–401.

[Mig83] A. A. Migdal. "Loop equations and $1/N$ expansion". In: *Physics Reports* 102.4 (1983), pp. 199–290.

[Mir+96] A. D. Mirlin et al. "Transition from Localized to Extended Eigenstates in the Ensemble of Power-Law Random Banded Matrices". In: *Physical Review E* 54.4 (1996), pp. 3221–3230.

[Mir07] M. Mirzakhani. "Simple Geodesics and Weil-Petersson Volumes of Moduli Spaces of Bordered Riemann Surfaces". In: *Inventiones mathematicae* 167.1 (2007), pp. 179–222.

[Mir41] C. Miranda. "Un'osservazione Su Un Teorema Di Brouwer". In: *Bollettino dell'Unione Matematica Italiana* 2.3 (1941), pp. 5–7.

[Mir97] R. Miranda. *Algebraic Curves and Riemann Surfaces.* Reprinted with corrections. Graduate Studies in Mathematics 5. Providence, RI: American Mathematical Society, 1997. 390 pp.

[MJD02] T. Miwa, M. Jimbo, and E. Date. *Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras.* Reprinted. Cambridge Tracts in Mathematics 135. Cambridge: Cambridge Univ. Press, 2002. 108 pp.

[MN04] J. A. Mingo and A. Nica. "Annular Noncrossing Permutations and Partitions, and Second-Order Asymptotics for Random Matrices". In: *International Mathematics Research Notices* 2004.28 (2004), pp. 1413–1460.

[MP22] P. Mergny and M. Potters. "Rank One HCIZ at High Temperature: Interpolating between Classical and Free Convolutions". In: *SciPost Physics* 12.1 (2022), p. 022.

[MS06] J. A. Mingo and R. Speicher. "Second Order Freeness and Fluctuations of Random Matrices: I. Gaussian and Wishart Matrices and Cyclic Fock Spaces". In: *Journal of Functional Analysis* 235.1 (2006), pp. 226–270.

[MS17] J. A. Mingo and R. Speicher. *Free Probability and Random Matrices.* Fields Institute Monographs. New York: Springer-Verlag, 2017.

[MŚS07] J. A. Mingo, P. Śniady, and R. Speicher. "Second Order Freeness and Fluctuations of Random Matrices: II. Unitary Random Matrices". In: *Advances in Mathematics* 209.1 (2007), pp. 212–240.

[MT01] B. Mohar and C. Thomassen. *Graphs on Surfaces.* Johns Hopkins University Press, 2001.

[MT61] N. F. Mott and W. D. Twose. "The Theory of Impurity Conduction". In: *Advances in Physics* 10.38 (1961), pp. 107–163.

[Mul84] M. Mulase. "Cohomological Structure in Soliton Equations and Jacobian Varieties". In: *Journal of Differential Geometry* 19.2 (1984).

[Mum07] D. Mumford. *Tata Lectures on Theta II.* Boston, MA: Birkhäuser Boston, 2007.

[Mum78] D. Mumford. "An Algebro-Geometric Construction of Commuting Operators and of Solutions to the Toda Lattice Equation, Korteweg deVries Equation and Related Nonlinear Equation". In: Proceedings of the International Symposium on Algebraic Geometry. 1978, pp. 115–153.

[Mum83] D. Mumford. *Tata Lectures on Theta I.* Red. by J. Oesterlé and A. Weinstein. Vol. 28. Progress in Mathematics. Boston, MA: Birkhäuser Boston, 1983.

[MW03] M. Mulase and A. Waldron. "Duality of Orthogonal and Symplectic Matrix Integrals and Quaternionic Feynman Graphs". In: *Communications in Mathematical Physics* 240.3 (2003), pp. 553–586.

[Nov10] J. Novak. "Jucys-Murphy Elements and the Unitary Weingarten Function". In: *Banach Center Publications* 1.89 (2010), pp. 231–235.

[Nov20] J. Novak. “On the Complex Asymptotics of the HCIZ and BGW Integrals”. 2020.

[Nov25] J. Novak. *Quasimodular Asymptotics of Spherical Integrals*. 2025. URL: http://arxiv.org/abs/2502.14056 (visited on 03/03/2025). Pre-published.

[Oko97] A. Okounkov. “Proof of a Conjecture of Goulden and Jackson”. In: *Canadian Journal of Mathematics* 49.5 (1997), pp. 883–886.

[Pas06] L. Pastur. “Limiting Laws of Linear Eigenvalue Statistics for Hermitian Matrix Models”. In: *Journal of Mathematical Physics* 47.10 (2006), p. 103303.

[Pol81] A. M. Polyakov. “Quantum Geometry of Bosonic Strings”. In: *Physics Letters B* 103.3 (1981), pp. 207–210.

[Pro76] C. Procesi. “The Invariant Theory of $n \times n$ Matrices”. In: *Advances in Mathematics* 19.3 (1976), pp. 306–381.

[Rey89] É. Reyssat. *Quelques aspects des surfaces de Riemann*. Progress in mathematics 77. Boston Basel Berlin: Birkhäuser, 1989. 166 pp.

[Rot64] G.-C. Rota. “On the Foundations of Combinatorial Theory I. Theory of Möbius Functions”. In: *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 2.4 (1964), pp. 340–368.

[Sam80] S. Samuel. “U(N) Integrals, 1/N, and the De Wit–’t Hooft Anomalies”. In: *Journal of Mathematical Physics* 21.12 (1980), pp. 2695–2703.

[Sch88] F. Schottky. “Zur Theorie der Abelschen Functionen von vier Variabeln.” In: *Journal für die reine und angewandte Mathematik* 102 (1888), pp. 304–352.

[Sch98] G. Schaeffer. “Conjugaison d’arbres et cartes combinatoires aléatoires”. Université Bordeaux I, 1998.

[Shc13] M. Shcherbina. “Fluctuations of Linear Eigenvalue Statistics of β Matrix Models in the Multi-cut Regime”. In: *Journal of Statistical Physics* 151.6 (2013), pp. 1004–1034.

[Shc14] M. Shcherbina. “Change of variables as a method to study general β-models: bulk universality”. In: *Journal of Mathematical Physics* 55.4 (2014), p. 043504.

[Shi86] T. Shiota. “Characterization of Jacobian Varieties in Terms of Soliton Equations”. In: *Inventiones Mathematicae* 83 (1986).

[Spa81] E. H. Spanier. *Algebraic Topology*. New York, NY: Springer, 1981.

[Spe83] T. P. Speed. “Cumulants and Partition Lattices”. In: *Australian Journal of Statistics* 25.2 (1983), pp. 378–388.

[Sta89] R. P. Stanley. “Some Combinatorial Properties of Jack Symmetric Functions”. In: *Advances in Mathematics* 77.1 (1989), pp. 76–115.

[Tai97] I. A. Taimanov. “Secants of Abelian Varieties, Theta Functions, and Soliton Equations”. In: *Russian Mathematical Surveys* 52.1 (1997), pp. 147–218.

[tH74] G. ’t Hooft. “A Planar Diagram Theory for Strong Interactions”. In: *Nuclear Physics B* 72.3 (1974), pp. 461–473.

[Tut62a] W. T. Tutte. “A Census of Hamiltonian Polygons”. In: *Canadian Journal of Mathematics* 14 (1962), pp. 402–417.

[Tut62b] W. T. Tutte. “A Census of Planar Triangulations”. In: *Canadian Journal of Mathematics* 14 (1962), pp. 21–38.

[Tut62c] W. T. Tutte. “A Census of Slicings”. In: *Canadian Journal of Mathematics* 14 (1962), pp. 708–722.

[Tut63] W. T. Tutte. “A Census of Planar Maps”. In: *Canadian Journal of Mathematics* 15 (1963), pp. 249–271.

[Tut68] W. T. Tutte. “On the Enumeration of Planar Maps”. In: *Bulletin of the American Mathematical Society* 74.1 (1968), pp. 64–75.

[Tut84] W. T. Tutte. *Graph Theory*. Encyclopedia of Mathematics and Its Applications 21. Menlo Park, Calif.: Addison-Wesley, 1984. 333 pp.

[vdHof16] R. van der Hofstad. *Random Graphs and Complex Networks*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge: Cambridge University Press, 2016.

[Voi90] D. Voiculescu. “Circular and Semicircular Systems and Free Product”. In: *Progr. Math.* 92 (1990), pp. 45–60.

[Voi91] D. Voiculescu. “Limit Laws for Random Matrices and Free Products”. In: *Inventiones mathematicae* 104.1 (1991), pp. 201–220.

[Voi99] D. Voiculescu. “The Analogues of Entropy and of Fisher’s Information Measure in Free Probability Theory: VI. Liberation and Mutual Free Information”. In: *Advances in Mathematics* 146.2 (1999), pp. 101–166.

[Wei78] D. Weingarten. “Asymptotic Behavior of Group Integrals in the Limit of Infinite Rank”. In: *Journal of Mathematical Physics* 19.5 (1978), pp. 999–1001.

[Wic50] G. C. Wick. “The Evaluation of the Collision Matrix”. In: *Physical Review* 80.2 (1950), pp. 268–272.

[Wig55] E. P. Wigner. “Characteristic Vectors of Bordered Matrices With Infinite Dimensions”. In: *Annals of Mathematics* 62.3 (1955), pp. 548–564.

[Wig58] E. P. Wigner. “On the Distribution of the Roots of Certain Symmetric Matrices”. In: *Annals of Mathematics* 67.2 (1958), pp. 325–327.

[Wig67] E. P. Wigner. “Random Matrices in Physics”. In: *SIAM Review* 9.1 (1967), pp. 1–23.

[Wis28] J. Wishart. “The Generalised Product Moment Distribution in Samples from a Normal Multivariate Population”. In: *Biometrika* 20A.1/2 (1928), pp. 32–52.

[WL72a] T. R. S. Walsh and A. B. Lehman. “Counting Rooted Maps by Genus II”. In: *Journal of Combinatorial Theory, Series B* 13.2 (1972), pp. 122–141.

[WL72b] T. R. S. Walsh and A. B. Lehman. “Counting Rooted Maps by Genus. I”. In: *Journal of Combinatorial Theory, Series B* 13.3 (1972), pp. 192–218.

[Zvo97] A. Zvonkin. “Matrix Integrals and Map Enumeration: An Accessible Introduction”. In: *Mathematical and Computer Modelling: An International Journal* 26.8–10 (1997), pp. 281–304.

[ZZ03] P. Zinn-Justin and J.-B. Zuber. “On Some Integrals over the U(N) Unitary Group and Their Large N Limit”. In: *Journal of Physics A: Mathematical and General* 36.12 (2003), pp. 3173–3193.

Résumé

Cette thèse traite de plusieurs problèmes reliés aux matrices aléatoires et à l'énumération de cartes. Informellement, les cartes sont des graphes dessinés sur des surfaces. Les travaux des physiciens Brézin, Itzkson, Parisi, et Zuber ont permis de comprendre que le problème de l'énumération des cartes est relié à la distribution des valeurs propres de matrices aléatoires Hermitiennes Gaussiennes. Ce lien, beaucoup étudié depuis, s'est révélé fructueux dans les deux directions: une bonne compréhension de la combinatoire des cartes permet de décrire le spectre de matrices aléatoires, et des méthodes analytiques applicables aux intégrales de matrices permettent d'approcher des problèmes combinatoires a priori difficiles. Le Chapitre 3 de cette thèse propose une description de modèles de matrices aléatoires unitaires en terme d'une famille de cartes, les cartes de type unitaire. Ces cartes constituent une généralisation d'une famille d'objets combinatoires liés aux probabilités libres, les nombres de Hurwitz monotones. D'autre part, la distribution de valeurs propres de matrices Hermitiennes unitairement invariantes est un cas particulier d'une famille de mesures appelée β-ensemble. Au Chapitre 4, on propose une méthode directe de calcul des moments du β-ensemble en terme de cartes. Cette approche propose un point de vue nouveau sur les moments du β-ensemble, différent de celui considéré par LaCroix, dans le cadre la b-conjecture de Goulden et Jackson en combinatoire algébrique.

Des relations clés pour étudier les liens entre cartes et matrices aléatoires sont les équations de Dyson-Schwinger. En théorie des matrices aléatoires, ces équations apparaissent comme conséquence de l'invariance par translation de la mesure de référence considérée. Au dela, ces équations apparaissent sous d'autres formes dans de nombreux domaines: en particulier, il s'agit des équations de Tutte en combinatoire des cartes. Elles peuvent être vues comme un cas particulier de la récurrence topologique de Eynard et Orantin. Ecrites pour le β-modèle et dans la limite de grande dimension, les équations de Dyson-Schwinger peuvent être interprétées comme définissant une courbe hyperelliptique, la courbe spectrale. De nombreuses observables de matrices aléatoires correspondent à des objets géométriques définis en terme de la courbe spectrale. Il est alors possible de réinterpréter des identités probabilistes de matrices aléatoires d'un point de vue géométrique. Une telle approche est discutée au Chapitre 5, pour obtenir des analogues Pfaffiens de la formule de Fay.

Plutôt que le spectre, on peut étudier les vecteur propres de matrices aléatoires. Dans ce cadre, on discute du problème de localisation des vecteurs propres d'une matrice d'adjacence d'un graphe aléatoire. Cette question est liée notamment au problème de la localisation d'Anderson, un problème encore partiellement ouvert en physique mathématique. Au Chapitre 6, on étudie le problème de localisation pour le modèle Generalized Random Graph, qui généralise le modèle d'Erdős-Rényi.